# Surgeries on periodic links and homology of periodic 3-manifolds

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*Abstract*

Fix a prime integer $p$. We show that a closed orientable 3-manifold $M$ admits an action of $\mathbf{Z}_p$ with fixed point set $S^1$ if and only if $M$ can be obtained as the result of surgery on a $p$-periodic framed link $L$ and $\mathbf{Z}_p$ acts freely on the components of $L$. We prove a similar theorem for free $\mathbf{Z}_p$-actions. As an interesting application, we prove the following, rather unexpected result: for any $M$ as above and for any odd prime $p$, $H_1(M, \mathbf{Z}_p) \neq \mathbf{Z}_p$. We also prove a similar criterion of 2-periodicity for rational homology 3-spheres.

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## 0. *Introduction*

In the early 1960's both Wallace [24] and Lickorish [7] proved that every closed, connected, orientable 3-manifold may be obtained by surgery on a framed link in $S^3$. Thus, link diagrams may be used to depict manifolds. Every manifold has infinitely many different framed link descriptions. However, in 1970's Kirby [6] showed that two framed links determine the same 3-manifold if and only if they are related by a finite sequence of two specific types of moves. This calculus of framed links, together with the earlier results, gives a classification of 3-manifolds in terms of equivalence classes of framed links. The framed link representation of 3-manifolds has proven to be extremely useful. For instance, most of the new 3-manifold invariants originating from famous Witten's paper [25] are based on the framed link approach. Therefore, it is always very useful to have some kind of correspondence between certain classes of 3-manifolds and some classes of framed links. One example of such correspondence is the classical relationship between the lens spaces and the chain-link diagrams (see, for instance, [19]). This result, in particular, allowed L. Jeffrey to determine exact formulas for the Witten-Reshetikhin-Turaev invariants of the lens spaces [4]. Another example is the fact that a closed oriented 3-manifold is an integral homology 3-sphere if and only if it can be obtained by surgery on an algebraically split link with framing numbers $\pm 1$ (see [10, 12]). This relationship plays a key role in many papers on quantum and finite invariants of integral homology 3-spheres (see, for instance, [12, 13, 10, 11]). In Section 1 we will establish an analogous relationship between periodic 3-manifolds and periodic links. Namely, we prove the following theorem:

THEOREM 1·1. *Let $p$ be a prime integer and $M$ be a closed oriented 3-manifold. There is an action of the cyclic group $\mathbf{Z}_p$ on $M$ with the fixed-point set equal to a circle if and only if there exists a framed $p$-periodic link $L \subset S^3$ such that $M$ is the result of surgery on $L$ and $\mathbf{Z}_p$ acts freely on the set of* *components of $L$.*

A special case of Theorem 1·1 when $M$ is a homology sphere was proven in [5]. In the general form the theorem was proven for the first time by the first author in his graduate course *Topics in Algebra Situs* (The George Washington University, February of 1999). A similar result is obtained for manifolds with free $\mathbf{Z}_p$ actions:

THEOREM 1·2. *Let $p$ be a prime integer and $M$ be a closed oriented 3-manifold. There is a free action of the cyclic group $\mathbf{Z}_p$ on $M$ if and only if there exists a framed $p$-periodic link $L \subset S^3$ admitting a free action of $\mathbf{Z}_p$ on the set of its components such that $M$ is the result of surgery on $L' = L \cup \gamma$, where $\gamma$ is the axis of the action with framing co-prime to $p$.*

In Section 2 we give an interesting application of Theorem 1·1. Namely, we prove the following result:

THEOREM 2·1. *If a closed orientable 3-manifold $M$ admits an action of a cyclic group $\mathbf{Z}_p$ where $p$ is an odd prime integer and the fixed point set of the action is $S^1$ then $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$.*

Note that this theorem provides a non-trivial criterion for 3-manifolds admitting the described action. The simplest examples of 3-manifolds with $H_1(M; \mathbf{Z}_p) = \mathbf{Z}_p$ are lens spaces $L_{pn,q}$ (or more generally, $(pn/q)$ Dehn surgeries on knots in $S^3$).

Theorem 2·1 was first announced as a conjecture[1] and partially proven,

[1]The conjecture was obtained as a result of extensive computations performed with a

in the case when the orbit space of the action can be obtained from $S^3$ by an integer surgery on a knot, in April of 1999 [21] (It is interesting to mention that the conjecture was influenced by the study of Murakami-Ohtsuki-Okada invariants on periodic 3-manifolds[2], but the equation $MOO_p(M) = \pm G_p^{rk H_1(M;\mathbf{Z}_p)}$ eventually led to the more "classical" algebraic topology. Here $p$ is an odd prime integer, $MOO_p$ is the Murakami-Ohtsuki-Okada invariant parameterized by $q = e^{2\pi i/p}$, and $G_p = \sum_{j\in \mathbf{Z}_p} q^{j^2}$ ). Recently (November, 1999), Adam Sikora announced a proof of the theorem [20]. In fact, using some classical but involved algebraic topology, he obtained more general results implying our theorem. The surgery presentation of periodic 3-manifolds developed in Section 1 allowed us to find an elementary proof of Theorem 2·1, presented in Section 2.

Theorem 2·1 is not true for $p = 2$ (see Remark 2·12). An interesting criterion for 2-periodic rational homology spheres is provided by the following theorem.

THEOREM 2·2. *Let $M$ be a rational homology 3-sphere such that the group $H_1(M;\mathbf{Z})$ does not have elements of order 16. If $M$ admits an orientation preserving action of $\mathbf{Z}_2$ with the fixed point set being a circle then the canonical decomposition of the group $H_1(M;\mathbf{Z})$ has even number of terms $\mathbf{Z}_2$ and even number of terms $\mathbf{Z}_4$, and arbitrary number of terms $\mathbf{Z}_8$.*

program written in *Mathematica*.

[2]In turn, our interest in Murakami-Ohtsuki-Okada invariants was sparked by their relation with the second skein module [18].

The second author thanks Yongwu Rong and Adam Sikora for useful conversations. When a preliminary version of the paper was ready, we received an e-mail from James Davis saying that he and his student Karl Bloch also found a proof for Theorem 2·1.

## 1. *Periodic 3-manifolds are surgeries on periodic links*

We show in this section that $p$-periodic closed oriented 3-manifolds can be presented as results of integer surgeries on $p$-periodic links. We show also an analogous result for manifolds with free action of $\mathbf{Z}_p$.

Before we prove theorems 1·1 and 1·2, we need to establish some basic terminology and preliminary lemmas.

### 1.1. *Periodic Links.*

*Definition.* By a *framed knot* $K$ we mean a ring $S^1 \times [0, \varepsilon]$ embedded in $S^3$. By the *framing* of $K$ we mean an integer defined as follows. Let $V_\varepsilon$ be the $\varepsilon$-neighborhood of $K_0 = S^1 \times \{0\}$, then $K_\varepsilon = S^1 \times \{\varepsilon\}$ is a projection of $K_0$ onto $\partial V_\varepsilon$. Let $P$ be the projection of $K_0$ onto $\partial V_\varepsilon$ which is homologically trivial in $S^3 - V_\varepsilon$. The framing $f$ is defined as the algebraic number of intersections of $K_\varepsilon$ and $P$. A *framed link* is a collection of non-intersecting framed knots. We will adopt the usual "blackboard" convention for framed link diagrams.

*Definition.* A (framed) link $L$ in $S^3$ is called *p-periodic* if there is a $\mathbf{Z}_p$-action on $S^3$, with a circle as a fixed point set, which maps $L$ onto itself, and such that $L$ is disjoint from the fixed point set. Furthermore, if $L$ is an oriented link, one assumes that each generator of $\mathbf{Z}_p$ preserves the orientation of $L$ or changes it to the opposite one.

By the positive solution of Smith Conjecture ([8, 23]) we know that the fixed point set of the action of $\mathbf{Z}_p$ is an unknotted circle and the action is conjugate to an orthogonal action on $S^3$. In other words, if we identify $S^3$ with $\mathbf{R}^3 \cup \infty$, then the fixed point set can be assumed to be equal to the "vertical" axis $z = 0$ together with $\infty$, and a generator $\varphi$ of $\mathbf{Z}_p$ can be assumed to be the rotation $\varphi(z,t) = (e^{2\pi i/p} \cdot z, t)$, where the coordinates on $\mathbf{R}^3$ come from the product of the complex plane and the real line $\mathbf{C} \times \mathbf{R}$. Thus, any (framed) $p$-periodic link $L^p$ may be represented by a $\varphi$-invariant diagram, *p-periodic diagram* (with framing parallel to the projection plane), see Fig. 1.

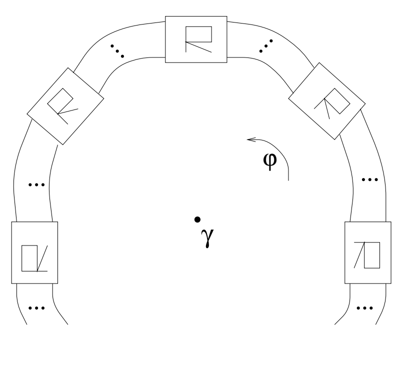


Fig. 1

By the *underlying link* for $L^p$ we will mean the orbit space of the action, that is $L_* = L^p/\mathbf{Z}_p$.

LEMMA 1·3. *Let $p$ be a prime integer and $L^p \subset S^3$ be a (framed) $p$-periodic link. The following three conditions are equivalent:*

*1) $\mathbf{Z}_p$ acts freely on the set of components of $L^p$;*

*2) The linking number of each component of the underlying link $L_*$ with the axis of rotation is congruent to zero modulo $p$;*

*3) The number of components of $L^p$ is $p$ times greater than the number of components of $L_*$.*

*Proof.* The equivalence of the conditions 1) and 3) is obvious. To prove that 2) is equivalent to 3), consider the covering projection $\rho: L^p \to L_*$. Let $l$ be a component of $L_*$. The preimage $\rho^{-1}$ of the closed path $\lambda$ which traverses $l$ exactly once (i.e., $l$ with a base point) consists of $p$ paths $\lambda_1, \ldots, \lambda_p$ in $L^p$. The condition $\mathrm{lk}(l,\gamma) = 0 \pmod p$ is equivalent to the condition that each of $\lambda_i$ is closed. Thus, $l$ lifts to $p$ components in $L^p$ if and only if $\mathrm{lk}(l,\gamma) = 0 \pmod p$. □

*Definition.* A $p$-periodic link $L^p$ that satisfies any of the conditions from Lemma 1·3 will be called *strongly p-periodic.*

1.2. *Periodic Manifolds.*

*Definition.* A 3-manifold $M$ is called *p-periodic* if it admits an orientation preserving action of the cyclic group $\mathbf{Z}_p$ with a circle as a fixed point set, and the action is free outside the circle.

We can immediately prove the easy part of Theorem 1·1. Indeed, consider a strongly $p$-periodic framed link $L^p$, and let $M$ be the 3-manifold obtained by surgery on $L^p$. By definition of a framed $p$-periodic link, there is a $\mathbf{Z}_p$ action on $S^3$, and on $S^3 - L^p$, with a circle $\gamma$ as a fixed point set. This action induces a $\mathbf{Z}_p$ action on $M$. Moreover, since the action of $\mathbf{Z}_p$ is free on the set of components of $L^p$, there are no other fixed points of the action of $\mathbf{Z}_p$ on $M$ but the circle $\gamma$.

To prove the difficult part of Theorem 1·1 we first fix some notation. Suppose that $\mathbf{Z}_p$ acts on $M$ with the fixed-point set equal to a circle $\gamma$. Denote the quotient by $M_* = M/\mathbf{Z}_p$, the projection map by $h\colon M \to M_*$ and $\gamma_* = h(\gamma)$.

LEMMA 1·4. *The map* $h_*\colon H_1(M) \to H_1(M_*)$ *is an epimorphism.*

*Proof.* Let $x_0 \in \gamma$. Since $x_0$ is a fixed point of the action, any loop based at $h(x_0)$ lifts to a loop based at $x_0$. Thus $h_\# : \pi_1(M, x_0) \to \pi_1(M_*, h(x_0))$ is an epimorphism, and since $H_1$ is an abelianization of $\pi_1$, the map $h_*$ is also an epimorphism. □

Notice that the proof works for any finite group action on a manifold with a non-empty fixed point set.

Let us recall the Lefschetz' duality theorem which we will use in our proof of Theorem 1·1. First some terminology: a compact connected $n$-dimensional manifold $M$ is called *R-oriented* for a commutative ring with identity $R$, if

$H_n(M, \partial M; R) = R$. In particular, any manifold is $\mathbf{Z}_2$-oriented, and an oriented manifold is $R$-oriented for any ring $R$. For a reference, see [22].

THEOREM 1·5. (Lefschetz) *Let $M$ be a compact $n$-dimensional, $R$-oriented manifold. Then there is an isomorphism $\tau : H^q(M; R) \to H_{n-q}(M, \partial M; R)$. Furthermore if $R$ is a PID (principal ideal domain) and $H_{q-1}(M; R)$ is free then $H^q(M; R) = Hom(H_q(M; R), R)$ and for $\alpha \in H^q(M; R)$ and $c \in H_q(M; R)$ one has: $\alpha(c) = alg(c, \tau(\alpha))$, where $alg(c, \tau(c)) \in R$ is the algebraic intersection number of $c$ and $\tau(\alpha)$ in $M$ ( $alg$: $H_q(M; R) \times H_{n-q}(M, \partial M; R) \to R$).*

We use Lefschetz' Theorem to show that the covering $h : M \to M_*$ is determined by a 2-chain whose boundary is a multiple of $\gamma_*$. Because we work with $q = 1$, then $H_{q-1}(M, R)$ is free and we can use the intersection number interpretation of the Lefschetz' Theorem.

LEMMA 1·6. *Let $M$ be a closed orientable $p$-periodic 3-manifold. With the notation as before, one has:*

*(1) $\gamma_* \equiv 0$ in $H_1(M_*, \mathbf{Z}_p)$.*

*(2) There is a 2-chain $C \in C_2(M_*, \mathbf{Z}_p)$ such that $\partial C \equiv m\gamma_*$ mod $p$ and the covering $h : (M - \gamma) \to (M_* - \gamma_*)$ is determined by the map $\phi_C : H_1(M_* - \gamma_*) \to \mathbf{Z}_p$ where $\phi_C(K)$ is the intersection number of $K$ with $C$ (i.e. for a 1-cycle $K \in M_* - \gamma_*$, $\phi_C(K) = alg(K, C)$ where*

*$alg(K,C)$ is the intersection number of $K$ with $C$, well defined mod $p$)*[3].

*In particular $\phi_C(\mu_*) = m$, where $\mu_*$ is a meridian of $\gamma_*$.*

*Proof.* To work with Lefschetz' Theorem we have to consider compact manifolds. Thus, instead of $M_* - \gamma_*$ we consider a homotopically equivalent compact manifold $\hat{M}_* = M_* - int(V_{\gamma_*})$, where $V_{\gamma_*}$ is a regular neighborhood of $\gamma_*$ in $M_*$. Similarly, let $V_\gamma = h^{-1}(V_{\gamma_*})$ be a $Z_p$-invariant regular neighborhood of $\gamma$ in $M$. Let also $\hat{M} = M - int(V_\gamma)$. Since $\hat{h}\colon \hat{M} \to \hat{M}_*$ is a regular covering, it is characterized by an epimorphism $\pi_1(\hat{M}_*) \to \pi_1(\hat{M}_*)/\pi_1(\hat{M}) = \mathbf{Z}_p$ (up to an automorphism of $\mathbf{Z}_p$). Thus, since $\mathbf{Z}_p$ is abelian, $\hat{h}$ is defined by an epimorphism $\phi : H_1(\hat{M}_*) \to \mathbf{Z}_p$, where $\phi$ is unique up to an automorphism of $\mathbf{Z}_p$. Let $\hat{C}$ be a 2-cycle representing the element of $H_2(\hat{M}_*, \partial\hat{M}_*; \mathbf{Z}_p)$ dual to the epimorphism $\phi$, that is, such that $alg(K,\hat{C}) \equiv \phi(K) \pmod{p}$ for any $K \in H_1(\hat{M}_*)$. We can assume that $\hat{C} \cap \partial\hat{M}_*$ is a collection of simple noncontractible curves in the torus $\partial\hat{M}_*$. Finally let $C$ be a 2-chain obtained from $\hat{C}$ by adding to $\hat{C}$ annuli connecting the components of $\hat{C} \cap \partial\hat{M}_*$ with $\gamma_*$ in $V_{\gamma_*}$. Thus, $C$ of part (2) is constructed.

Let $\mu_*$ be a meridian of $\gamma_*$ (or more precisely, of $V_{\gamma_*}$). $\phi(\mu_*) \neq 0 \pmod{p}$ because $\gamma_*$ is a branching set of the covering, so the preimage of $\mu_*$ (under $h$) is a connected curve $\mu$, (a meridian of $\gamma$ in $M$), by the definition of a branched covering. Thus, there is $0 < m < p$ such that $\phi(\mu_*) = m$. We can conclude also that $m\gamma_* \equiv \partial C$ mod $p$, thus, since $p$ is prime, $\gamma_* \equiv 0$ in

[3] If $H_2(M_*, \mathbf{Z}) = 0$, then $alg(K,C) = lk(K, m\gamma_*)$, but generally $alg(K,C)$ depends on the choice of $C$.

$H_1(M_*; \mathbf{Z}_p)$. $\square$

By the *core* of the surgery we mean the framed surgery link, that is the framed link, regular neighborhood of which is removed in the "drilling" part of the surgery, with framing determined by the meridian of the attached ("filling" part of the surgery) solid torus. The *co-core* of the surgery is the core of the "filling" solid torus, with its framing determined by the meridian of the removed solid torus. The surgery on the co-core link brings back the initial manifold.

PROOF OF THEOREM 1·1. By the classical result of Wallace (1960) and Lickorish (1962), every closed oriented 3-manifold is a result of a surgery on a framed link in $S^3$. In particular, $M_*$ can be represented as a result of surgery on some framed link $L_\#$ in $S^3$. Inversely, $S^3$ can be obtained as a result of surgery on some framed link $\hat{L}_\#$ in $M_*$. We can assume that $\hat{L}_\#$ satisfies the following conditions (possibly after deforming $\hat{L}_\#$ by ambient isotopy):

(1) $\gamma_* \cap \hat{L}_\# = \emptyset$;

(2) $alg(\hat{L}^i_\#, C) \equiv 0 \ mod \ p$, for any component $\hat{L}^i_\#$ of $\hat{L}_\#$.

$\hat{L}_\#$ satisfying the conditions 1-2 can be obtained as follows:
Let $L_\# \subset S^3$ be a framed link in $S^3$ such that $M_*$ is a result of surgery on $L_\#$. Let $\hat{L}_\#$ denote the co-core of the surgery. In particular, $\hat{L}_\#$ is a framed link in $M_*$ such that $S^3$ is a result of surgery on $\hat{L}_\#$. By a general position argument, we can make $\gamma_*$ and $\hat{L}_\#$ disjoint, but in order to get condition (2)

we should do so in a controllable manner.

Let $C$ be the 2-chain from Lemma 1·6. Let $\hat{L}^i_\#$ be any component of $\hat{L}_\#$. If we change a crossing between $\hat{L}^i_\#$ and $\gamma_*$ then the algebraic crossing number, $alg(\hat{L}^i_\#, C)$ changes by $\pm m$ mod $p$. Thus, by a series of crossing changes we can get $alg(\hat{L}^i_\#, C) \equiv 0 \ mod \ p$ for any component of $\hat{L}_\#$, providing condition (2).

This implies that we can easily modify $C$ (outside $\gamma_*$) so that $C \cap \hat{L}_\# = \emptyset$. Therefore, $C$ survives the surgery (as well as $\gamma_*$), and in $S^3$ it has $\mathbf{Z}_p$ boundary $m\gamma_*$ and it is disjoint from $L_\#$ (link in $S^3$ being the co-core of the surgery on $\hat{L}_\#$ in $M_*$). Thus, $\mathrm{lk}(L^i_\#, \gamma_*) \equiv 0 \ mod \ p$ for any component $L^i_\#$ of $L_\#$.

Now we are ready to unknot $\gamma_*$ using Kirby calculus ([6], [2]). Choose some orientation on $\gamma_*$. We can add unlinked components with framing $\pm 1$ to $L_\#$ around each crossing of $\gamma_*$, making sure that arrows on $\gamma_*$ run opposite ways (i.e. the linking number of $\gamma_*$ with the new component is zero, see Fig. 2). Use the K-move to change the appropriate crossings and thus to unknot $\gamma_*$. Thus we trivialize $\gamma_*$ without compromising conditions (1) and (2). Denote the framed link obtained from the initial link $L_\#$ after the described isotopy and adding the new components by $L_*$. Notice, that each new component that we introduce during the above procedure has linking number 0 with any other component of $L_*$.

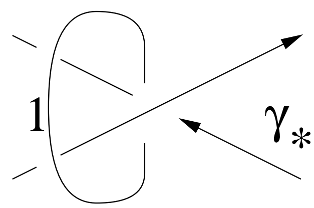


Fig. 2

To complete the proof of the theorem, consider the $p$-fold cyclic branched covering of $S^3$ by $S^3$ with branching set $\gamma_*$. Let $L$ denote the preimage of $L_*$. Notice that $L$ is strongly $p$-periodic. We claim that the result of performing surgery on $L$ is $\mathbf{Z}_p$-homeomorphic to $M$.

The preimage of each component of $L_*$ consists of $p$ components permuted by a $\mathbf{Z}_p$ action, by Lemma 1·4. Therefore, $\mathbf{Z}_p$ acts on the result of the surgery on $S^3$ along $L$, $\widetilde{M} = (S^3, L)$, with a branched set $\gamma$ and quotient $M_* = (S^3, L_*)$.

The group $H_1(S^3 - \gamma_* - L_*)$ is generated by $\mu_*$ (a meridian of $\gamma_*$) and meridians of the components of $L_*$, say $b_1, \ldots, b_k$. Therefore, the groups $H_1(M_* - \gamma_* - \hat{L}_\#)$ and $H_1(M_* - \gamma_*)$ are also generated by $\mu_*, b_1, \ldots, b_k$. By Lemma 1·6, the covering $\rho\colon (M - \gamma) \to (M_* - \gamma_*)$ is characterized by the map $\phi_C\colon H_1(M_* - \gamma_*) \to \mathbf{Z}_p$ (up to an automorphism of $\mathbf{Z}_p$), where $\phi_C(K)$ is the intersection number of $K \in H_1(M_* - \gamma_*)$ with $C$ modulo $p$. Similarly, the covering $\widetilde{\rho}\colon (\widetilde{M} - \gamma) \to (M_* - \gamma_*)$ is characterized by a map $\phi_2\colon H_1(M_* - \gamma_*) \to \mathbf{Z}_p$. By our construction, $\phi_C(\mu_*) = m$ and $\phi_C(b_i) = 0$ for every $1 \le i \le k$. We need to show that $\phi_2(\mu_*) = m'$ for some $m'$ coprime to

$p$, and $\phi_2(b_i) = 0$ for every $1 \leq i \leq k$. This follows from the fact that $\widetilde{\rho}^{-1}(b_i)$ consists of $p$ loops and $\widetilde{\rho}^{-1}(\mu_*)$ is a single loop. Thus, $\phi_2$ and $\phi_C$ are equivalent up to the automorphism of $\mathbf{Z}_p$ sending $m'$ to $m$. Therefore, the manifolds $(\widetilde{M} - \gamma)$ and $(M - \gamma)$ are $\mathbf{Z}_p$-homeomorphic, with a homeomorphism given by $g \colon (\widetilde{M} - \gamma) \to (M - \gamma)$ such that $\widetilde{\rho} = \rho \circ g$. Notice, that $\widetilde{M}$ can be obtained from $(\widetilde{M} - \gamma)$ by attaching a 2-handle along $\widetilde{\rho}^{-1}(\mu_*)$ and then a 3-handle, and $M$ can be obtained from $(M - \gamma)$ by attaching a 2-handle along $\rho^{-1}(\mu_*)$ and then a 3-handle. Since $g(\widetilde{\rho}^{-1}(\mu_*)) = \rho^{-1}(\mu_*)$, the homeomorphism $g$ can be extended to a $\mathbf{Z}_p$-homeomorphism $\hat{g} \colon \widetilde{M} \to M$. Our proof of the Theorem 1·1 is completed. □

*Remark* 1·7. Notice, that if in the proof of Theorem 1·1 we assumed that the link $L_\#$ was algebraically split then the link $L_*$ would be algebraically split as well. This remark will be important later in the proofs of Theorems 2.1 and 2.2.

*Remark* 1·8. The referee suggested the following alternative way to construct the link $\hat{L}_\#$ satisfying the conditions (1) and (2): *We will work modulo $p$ throughout. Let $L_\# \in S^3$ be as in the proof and we assume that $\gamma_* \cap L_\# = \emptyset$. Denote by $A = (a_{ij})$ the linking matrix of $L_\#$, and by $G = (g_j)$ the column vector, such that $g_j = lk(\gamma_*, L^j_\#)$. Note that the group $H_1(M_*; \mathbf{Z}_p)$ is generated by the meridians of $L_\#$, $m_1, \ldots, m_n$, and $A$ is the presentation matrix for $H_1(M_*; \mathbf{Z}_p)$ in that generating set. Moreover, $\gamma_* = g_1 m_1 + \ldots + g_n m_n$, as an element of $H_1(M_*; \mathbf{Z}_p)$. Then $\gamma_* = 0$ in $H_1(M_*; \mathbf{Z}_p)$ means that $G$ is* *in the image of $A$, that is, $G$ is a linear combination of column vectors $A_i$, where $A_i$ is the ith column of $A$. Therefore handle slidings (of $\gamma_*$ over $L_\#$) respecting the linear combination mentioned above change $\gamma_*$ to $\gamma'_*$ such that for every $j = 1, \dots, n$, $lk(\gamma'_*, L^j_\#) = 0$.*

As a corollary to Theorem 1·1 we obtain a proof of Theorem 1·2.

Proof of Theorem 1·2. Consider any $\mathbf{Z}_p$ equivariant knot, say $\hat{\gamma}$ in $M$. Let $V_{\hat{\gamma}}$ be a $\mathbf{Z}_p$ equivariant regular neighborhood of $\hat{\gamma}$ in $M$ and $\gamma'$ a curve on $\partial V_{\hat{\gamma}}$ which is also $\mathbf{Z}_p$ equivariant. Notice, that $\gamma'$ intersects a meridian of $V_{\hat{\gamma}}$ exactly once. Now let $M'$ be a manifold obtained from $M$ by a surgery on $\hat{\gamma}$ with the framing defined by $\gamma'$. Let $\gamma \in M'$ be the co-core of the surgery. The $\mathbf{Z}_p$ action on $M$ yields the action on $M'$ and our choice of framing guarantees that $\gamma$ is the (only) fixed point set of the action. Thus, we can apply to $M'$ the previous theorem. This proves that $M$ can be obtained by an integer surgery on $L \cup \gamma$, where $L$ is a strongly $p$-periodic link. Furthermore, the framing of $\gamma$ must be coprime to $p$, to insure that $\mathbf{Z}_p$ acts on the resulting manifold with no fixed points. □

*Remark* 1·9. We plan to extend Theorem 1·1 to any $\mathbf{Z}_p$ orientation preserving action on a closed 3-manifold $M$, and to $\mathbf{Z}_{p^k}$ actions.

## *2. Homology of periodic 3-manifolds.*

The main goal of this section is to give elementary proofs of Theorems 2.1

and 2.2 using the surgery presentation of $p$-periodic 3-manifolds developed in Section 1.

2.1. *Linking matrices of framed strongly $p$-periodic links and of algebraically split links.*

Let $L$ be a framed oriented link of $n$ components $l_1, \ldots, l_n$. The *linking matrix* of $L$ is the matrix $(a_{ij})_{n\times n}$ defined by

$$a_{ij} = \begin{cases} \mathrm{lk}(l_i, l_j) & \text{if } i \neq j \\ \text{framing of } l_i & \text{if } i = j \end{cases}$$

Let $L^p$ be a framed strongly $p$-periodic link and $L_*$ be the corresponding underlying link. Fix an orientation of $L_*$ and denote the components of $L_*$ by $l_1, \ldots, l_n$. Consider a $p$-periodic diagram of $L^p$. Denote the $p$ copies of the tangle $R$ from the diagram (see Fig. 1) by $R_1, \ldots, R_p$ in the clockwise order. Lift the orientation of $L_*$ to $L^p$. By Lemma 1·3, each component $l_i$ of $L_*$ has $p$ covering preimages in $L^p$. Denote them in a clockwise order by $l_{i1}, \ldots, l_{ip}$. By the clockwise order here we mean such an order that if we choose any point $x \in l_{ij} \cap R_j$ then the corresponding point in $R_{j+1}$ will belong to $l_{ij+1}$ (subscripts are treated modulo $p$).

Now consider the following natural order for the components of $L^p$:

$$l_{11}, \ldots, l_{1p}, l_{21}, \ldots, l_{2p}, \ldots, l_{n1}, \ldots, l_{np}.$$

It is not hard to see that with regard to this order, the linking matrix $A_p$ for

$L^p$ is of the following form

$$A_p = \begin{pmatrix} A_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & A_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & A_{nn} \end{pmatrix},$$

where all the blocks are $p \times p$, $A_{ii}$ is the linking matrix for the sublink consisting of $l_{i1}, \ldots, l_{ip}$, and $B_{ij}$ is the matrix with elements $b^{ij}_{ks} = \mathrm{lk}(l_{ik}, l_{js})$.

Recall, that a matrix $(a_{ij})_{k\times k}$ is called *circulant* if $a_{ij} = a_{i+1j+1}$, $i, j = 1, \ldots, k$ (subscripts mod $k$).

PROPOSITION 2·3. *Every block in $A_p$ is a circulant matrix.*

*Proof.* Consider $B_{ij} = (b^{ij}_{ks})_{p\times p}$. Then $b^{ij}_{ks} = \mathrm{lk}(l_{ik}, l_{js})$ and $b^{ij}_{k+1s+1} = \mathrm{lk}(l_{ik+1}, l_{js+1})$. If one rotates the $p$-periodic diagram of $L^p$ around the center in the clockwise direction by $2\pi/p$ then the pair $(l_{ik}, l_{js})$ will go into $(l_{ik+1}, l_{js+1})$, taking subscripts modulo $p$. Thus, $\mathrm{lk}(l_{ik}, l_{js}) = \mathrm{lk}(l_{ik+1}, l_{js+1})$. If we notice that the framing numbers of $l_{i1}, \ldots, l_{ip}$ are all the same, then the above argument shows that $A_{ii}$ is also circulant for any $i = 1, \ldots, n$. □

*Definition.* We will call a (framed) link $L$ *algebraically split* if the linking number between any two components of $L$ is zero. A strongly $p$-periodic (framed) link $L^p$ will be called *orbitally separated* if the underlying link $L_*$ is algebraically split.

*Remark* 2·4. It is not hard to see that a strongly $p$-periodic link $L^p$

is orbitally separated if and only if any two components of $L^p$ that cover different components of $L_*$ have the linking number equal to zero.

COROLLARY 2·5. *It follows from Proposition 2·3 and Remark 2.4 that* $L^p$ *is an orbitally separated link if and only if all the non-diagonal blocks* $B_{ij}$ *in* $A_p$ *are zero matrices.* □

2.2. *Nullity of symmetric circulant matrices over* $\mathbf{Z}_p$.

Circulant matrices are very well studied and a lot is known about them (see, for instance, [1]). But, apparently, not much is known about circulant matrices over finite fields (or rings). The following two results provide the key tool for our proof of Theorem 2·1, but they also appear to be interesting from a purely matrix theoretical point of view.

LEMMA 2·6. *Let* $p$ *be a prime integer and let*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_p \\ a_p & a_1 & a_2 & \cdots & a_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$$

*be a circulant matrix with integer elements, then*

$$det\ A = a_1^p + a_2^p + \ldots + a_p^p \pmod{p}.$$

*Proof.* The determinant of $A$ is a sum of $p!$ terms. The terms of the form $a_i^p$, $i = 1, \ldots, p$, will be called *diagonal*. Note that any term different from diagonal appears in the sum exactly $p$ times:

$$a_{i_1} a_{i_2} a_{i_3} \ldots a_{i_p},$$

$$a_{i_2} a_{i_3} \ldots a_{i_p} a_{i_1},$$

$$\vdots$$

$$a_{i_p} a_{i_1} a_{i_2} \ldots a_{i_{p-1}}.$$

Note that the sign for all such terms is the same. To prove this we need to show that the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ i_1 & i_2 & i_3 & \cdots & i_p \end{pmatrix} \text{ and } \sigma' = \begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ i_p + 1 & i_1 + 1 & i_2 + 1 & \cdots & i_{p-1} + 1 \end{pmatrix}$$

have the same parity (everything is modulo $p$). Obviously,

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ i_p + 1 & i_1 + 1 & i_2 + 1 & \cdots & i_{p-1} + 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & \cdots & p & 1 \\ i_1 + 1 & i_2 + 1 & \cdots & i_{p-1} + 1 & i_p + 1 \end{pmatrix}.$$

The row $(\,2 \quad 3 \quad 4 \quad \cdots \quad p \quad 1\,)$ has $p - 1$ inversions. The numbers of inversions in $(\,i_1 \quad i_2 \quad \cdots \quad i_p\,)$ and $(\,i_1 + 1 \quad i_2 + 1 \quad \cdots \quad i_p + 1\,)$ also differ by

$p-1$. Thus, the parities of $\sigma$ and $\sigma'$ are the same. Therefore, the total sum of all non-diagonal terms in det $A$ is 0 mod $p$. The result follows. $\square$

Denote the nullity of $A$ over $\mathbf{Z}_n$ by $null_n A$.

LEMMA 2·7. *Let $p$ be an odd prime integer and $A$ be a $p \times p$ symmetric circulant matrix over $\mathbf{Z}_p$, then $null_p A \neq 1$.*

*Proof.* Assume that $null_p A > 0$. Since $A$ is a symmetric circulant matrix over $\mathbf{Z}_p$ and $p$ is an odd prime integer, by Lemma 2·6 we have

$$detA = a_1^p + 2a_2^p + \ldots + 2a_{\frac{p+1}{2}}^p = 0 \pmod{p}.$$

Therefore, by Fermat's theorem,

$$a_1 + 2a_2 + \ldots + 2a_{\frac{p+1}{2}} = 0 \pmod{p}.$$

After adding all rows to the last one and all columns to the last column we get

$$det\ A = det \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_3 & 0 \\ a_2 & a_1 & a_2 & \cdots & a_4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_3 & a_4 & a_5 & \cdots & a_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Denote the $i$th column of the above matrix by $C_i$. Then the linear combination

$$(p-1)C_1 + (p-2)C_2 + \ldots + 2C_{p-2} + C_{p-1}$$

is 0 modulo $p$. Indeed, the $i$th row of the linear combination is

$$(p-1)a_i + (p-2)a_{i+1} + \ldots + 2a_{i-3} + a_{i-2},$$

all the coefficients and subscripts are modulo $p$. It is not hard to see that after the substitution $a_1 = -2a_2 - 2a_3 - \ldots - 2a_{\frac{p+1}{2}}$, the coefficient for $a_k$ $(k \neq 1)$ in the above sum is

$$-2(p-i) + (p-(i+k)) + (p-(i-k)) = 0.$$

Thus, if $det\ A = 0$ over $\mathbf{Z}_p$ then $null_p A \geq 2$. □

*Remark* 2·8. Lemma 2·7 is not true if $p = 2$. For instance, nullity of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is 1 over $\mathbf{Z}_2$.

2.3. *Proof of Theorem* 2·1.

In this section we will prove Theorem 2·1, the main theorem of Section 2. Let $M$ be a closed oriented 3-manifold obtained by a Dehn surgery on a framed oriented link $L$, and let $A$ be the linking matrix of $L$. The following fact is well-known.

LEMMA 2·9. $null_p A = rank\ H_1(M; \mathbf{Z}_p)$. □

Now we are ready to prove an important special case of Theorem 2·1.

PROPOSITION 2·10. *Let $p$ be an odd prime integer. If a closed orientable 3-manifold $M$ can be obtained from $S^3$ by Dehn surgery on an orbitally separated framed link $L^p$ then $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$.*

*Proof.* Assume that $M$ can be obtained by Dehn surgery on an orbitally separated framed link $L^p$. Let $A_p$ be the linking matrix of $L^p$, as constructed in Section 2.1. By Corollary 2·5, $A_p$ is a block diagonal matrix. Therefore, $null_p A_p$ is equal to the sum $null_p A_{11} + \ldots + null_p A_{nn}$. By Proposition 2·3, each $A_{ii}$ is a circulant matrix. Moreover, since $A_p$ is a linking matrix, each $A_{ii}$ is symmetric. By Lemma 2·7, $null_p A_{ii} \neq 1$, hence, $null_p A_p \neq 1$. It follows from Lemma 2·9 that $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$. □

To finish our proof of the main theorem of Section 2 we need the following result ([11, Corollary 2·3]; see also [14, Corollary 2·5], for a somewhat stronger result).

PROPOSITION 2·11 (H. Murakami) *Fix an odd prime $r$. For every connected, closed, oriented 3-manifold $M$, there exist lens spaces $L(n_1, 1), \ldots, L(n_k, 1)$ with $n_i$ coprime to $r$ such that the connected sum $M \# L(n_1, 1) \# \ldots \# L(n_k, 1)$ can be obtained by Dehn surgery on an algebraically split link with integer framing.* □

Now we are ready to prove Theorem 2·1 in full generality.

PROOF OF THEOREM 2·1. Let $M$ be a $p$-periodic closed oriented 3-

manifold and $M_* = M/\mathbf{Z}_p$. By Proposition 2·11, there are integers $n_1, \ldots, n_k$ coprime to $p$ such that the connected sum $\widetilde{M_*} = M_* \# L(n_1, 1) \# \ldots \# L(n_k, 1)$ can be obtained by Dehn surgery on an algebraically split framed link $\widetilde{L_*}$. Consider $\widetilde{M} = M \# pL(n_1, 1) \# \ldots \# pL(n_k, 1)$. Obviously, $\widetilde{M}$ is $p$-periodic such that $\widetilde{M_*} = \widetilde{M}/\mathbf{Z}_p$. Moreover, it easily follows from the proof of Theorem 1·1 that $\widetilde{M}$ can be obtained using Dehn surgery on an orbitally separated framed link (see Remark 1·7). Therefore, by Proposition 2·10, $H_1(\widetilde{M}; \mathbf{Z}_p) \neq \mathbf{Z}_p$. Since the numbers $n_1, \ldots, n_k$ are coprime to $p$, we have $H_1(L(n_i, 1), \mathbf{Z}_p) = 0$ for every $i$. This implies that $H_1(M; \mathbf{Z}_p) = H_1(\widetilde{M}; \mathbf{Z}_p) \neq \mathbf{Z}_p$. □

2.4. *Orientation preserving $\mathbf{Z}_2$ actions. Proof of Theorem 2·2.*

*Remark* 2·12. Theorem 2·1 is not true in the case $p = 2$. A simple counterexample is $S^2 \times S^1$. It is interesting to notice that $S^2 \times S^1$ admits two different orientation preserving actions of $\mathbf{Z}_2$ with the fixed point set being a circle. Indeed, let $H(1,1)$ be the negative Hopf link with framing 1 on each component and let $H(-1,-1)$ be the negative Hopf link with framing $-1$ on each component (see Fig. 3). Dehn surgery on each of these framed links produces $S^2 \times S^1$. Moreover, permutation of the components defines $\mathbf{Z}_2$ actions on $S^2 \times S^1$ with a circle as the fixed point set. These two actions are different. In the first case the orbit space of the action $M_*$ is $S^2 \times S^1$, and in the second case $M_* = RP^3$. Furthermore, one can see that in the first case $\mathbf{Z}_2$ acts on $\mathbf{Z} = H_1(S^2 \times S^1)$ trivially, and in the second case it sends 1 to $-1$. Both of the described actions were studied in [17].

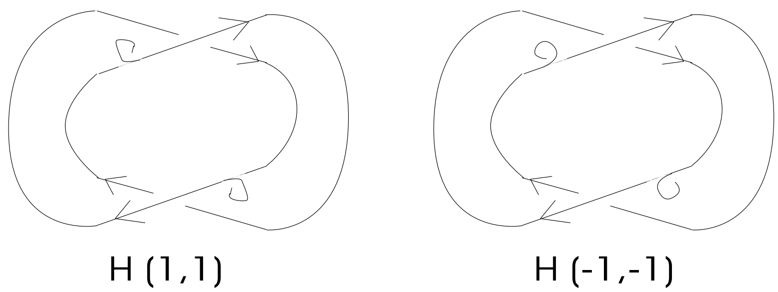


Fig. 3

An interesting criterion for 2-periodic rational homology 3-spheres is provided by Theorem 2·2. Before we prove it, let us recall that every finite abelian group can be uniquely decomposed into a direct sum of cyclic groups whose orders are powers of prime numbers. Such decomposition will be called *canonical.*

Proof of Theorem 2·2. Let $M$ be a rational homology 3-sphere such that $H_1(M, \mathbf{Z})$ does not have elements of order 16. Assume that $M$ admits an orientation-preserving action of $\mathbf{Z}_2$ such that the fixed point set is a circle. As before, let $M_* = M/\mathbf{Z}_2$ be the orbit space of the action. By Theorem 1·1, $M$ can be obtained using Dehn surgery on a strongly 2-periodic framed link $L^2$ with the underlying link $L_*$. By Lemma 1·4, $M_*$ is also a rational homology 3-sphere. Therefore, $M_*$ can be obtained by surgery on an algebraically split framed link (see [10, 12]). Thus, we may assume that $L_*$ is algebraically split (see Remark 1·7). By definition $L^2$ is orbitally separated, and by Proposition 2·3 and Corollary 2·5 the linking matrix of $L^2$ is block diagonal with every block being a $2 \times 2$ symmetric circulant matrix. Recall that the linking matrix of $L^2$ can be considered as a presentation matrix for the abelian group $H_1(M, \mathbf{Z})$. Therefore, it is enough to show that every finite abelian group $G$ presented by a matrix $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a, b \in \mathbf{Z}$ either has an element of order 16 or there are even number of terms $\mathbf{Z}_2$ and even number of terms $\mathbf{Z}_4$ in the canonical decomposition of $G$, moreover it is possible that the canonical decomposition of $G$ contains only one term of the form $\mathbf{Z}_{2^t}$ for any $t \geq 3$. Denote by $g$ the greatest common divisor of $a$ and $b$. Since $G$ is finite, we have $detA \neq 0$ and $g > 0$. It is easy to see that $G$ can be presented by the diagonal matrix $\widetilde{A} = \begin{pmatrix} g & 0 \\ 0 & \frac{|detA|}{g} \end{pmatrix}$. Therefore $G \simeq \mathbf{Z}_g \oplus \mathbf{Z}_{|detA|/g}$ (here by $\mathbf{Z}_1$ we mean the trivial group).

If $a$ and $b$ are both odd then $g$ is odd and $\frac{|detA|}{g}$ is divisible by 8.

If $a$ and $b$ are both even then $g$ is even. Let $t$ be the power of 2 in the prime decomposition of $g$. We have two different cases: 1) $\frac{|detA|}{g^2}$ is odd. Then $G \simeq \mathbf{Z}_{g/2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2l+1}$, where $2l + 1 = \frac{|detA|}{g2^t}$. 2) $\frac{|detA|}{g^2}$ is even. This means that both $\frac{a}{g}$ and $\frac{b}{g}$ are odd and therefore $\frac{|detA|}{g^2}$ is divisible by $2^s$, $s \geq 3$, which implies that $G \simeq \mathbf{Z}_{g/2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2^{s+t}} \oplus \mathbf{Z}_{2h+1}$, where $2h + 1 = \frac{|detA|}{g2^{s+t}}$. In this case we have an element of order $2^{s+t}$, which is at least 16.

We are left with the case when one of the numbers $a$ and $b$ is even and the other one is odd. But in this case $g$ is odd and $|detA|$ is odd.

Therefore, if $G$ does not have elements of order 16 then the number of terms $\mathbf{Z}_2$ and the number of terms $\mathbf{Z}_4$ in the canonical decomposition of $G$

are both even numbers. To see that it is possible to have a single $\mathbf{Z}_8$ term, consider the case $a = 3$ and $b = 1$ (An example of a manifold with such first homology is the lens space $L(8, 3)$). □

COROLLARY 2·13 *Let $M$ be a rational homology 3-sphere such that the group $H_1(M; \mathbf{Z})$ does not have elements of order 8. If $M$ admits an orientation preserving action of $\mathbf{Z}_2$ with the fixed point set being a circle then $H_1(M; \mathbf{Z}_2) \simeq \mathbf{Z}_2^{\oplus m}$ for some even integer $m$.*

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