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QUANTUM INVARIANTS, SKEIN MODULES, AND PERIODICITY OF 3-MANIFOLDS

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ABSTRACT

We study various quantum invariants and skein modules of 3-manifolds and their relationship to each other and to classical invariants of 3-manifolds. We study, in particular, how quantum invariants and classical invariants (homology) reflect periodicity of 3-manifolds. The first chapter is devoted to Turaev-Viro invariants. We show that each Turaev-Viro invariant is a sum of three invariants, study properties of the summand invariants and establish certain relationships. We solve a conjecture due to Kauffman and Lins and provide a simple and effective criterion answering whether two given lens spaces are distinguished by Turaev-Viro invariants or not. Chapter 2 is probably the most interesting chapter of the theses. We develop a special surgery presentation for p-periodic 3-manifolds and prove the following result: *if a closed orientable 3-manifold M admits an action of a cyclic group $\mathbf{Z}_p$ where p is an odd prime integer and the fixed point set of the action is S^1 then $H_1(M;\mathbf{Z}_p) \neq \mathbf{Z}_p$.* We also provide a similar criterion for the 2-periodic rational homology 3-spheres. Chapter 3 studies how quantum invariants reflect periodicity of 3-manifolds. We consider Dijkgraaf-Witten invariants and the simplest known quantum invariants discovered by H. Murakami, T. Ohtsuki, and M. Okada. In the last chapter, Chapter 4, we study some simple skein modules ($\mathcal{S}$ and $\mathcal{S}_2$) and their relationship with quantum invariants. We show how to construct Murakami-Ohtsuki-Okada invariants from these skein modules using Lickorish's method. In Section 3 of Chapter 4 we will show that if q is a root of unity then no other invariants can be obtained using Lickorish method on $\mathcal{S}$ and $\mathcal{S}_2$. We also study Deloup invariants and their relationship with the tensor product of two copies of $\mathcal{S}_2(M;R,q)$. In Appendix A we show how to define Dijkgraaf-Witten invariants using singular triangulations and simple 2-skeletons of 3-manifolds. In Appendix B we present tables with values of Turaev-Viro invariants for 239 closed oriented 3-manifolds of small complexity. The values are presented as cyclotomic integers. In Appendix C we show how to determine linking pairing of a rational homology 3-sphere from its rational surgery presentation.

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INTRODUCTION

Topology of 3-dimensional manifolds is a reach and interesting subject, and until now there is no definite way to decide whether two arbitrary chosen 3-manifolds are topologically equivalent (homeomorphic) or not. One of the possible ways to solve this problem is to find a sufficient set of invariants for 3-manifolds—functions on homeomorphic classes of 3-manifolds with values in more familiar algebraic structures, such as fields, rings, groups, modules. Thus, every discovery of a new 3-manifold invariant is very important. An astonishing breakthrough in this area happened in the end of 80s, when E. Witten outlined a new way for constructing 3-manifold invariants with values in complex numbers [Wi]. First realizations of Witten's ideas carried out by Reshetikhin and Turaev [RT], Turaev and Viro [TV], Dijkgraaf and Witten [DW] proved that the new approach was extremely productive. The new invariants appeared to be different from any other invariants previously known. Some of Witten's techniques and methods came from quantum physics, that is one of the reasons why the new invariants are sometimes called quantum invariants of 3-manifolds.

Another type of 3-manifold invariants are skein modules, introduced independently by J. Przytycki in 1987 and V. Turaev in 1988 (see [Pr1] and [Pr2] for an overview). These invariants are similar in nature to homology groups, but in general are much stronger, and much harder to compute. It turned out that the quantum invariants and skein modules are related in several very intriguing ways. In particular, most of quantum invariants can be defined using skein theoretical language. The first result in this direction is due to Lickorish ([Li2], [Li3], [Li4]) who constructed Reshetikhin-Turaev invariants using Kauffman bracket skein module.

In this thesis we will study various quantum invariants and skein modules of 3-manifolds and their relationship to each other and to classical invariants of 3-manifolds. We will study, in particular, how quantum invariants and classical invariants (homology) reflect periodicity of 3-manifolds.

The first chapter is devoted to Turaev-Viro invariants. These invariants can be defined using convenient and natural presentations of 3-manifolds, such as simple and special spines, triangulations and Heegaard diagrams. This allowed us to notice and prove a few interesting properties of the invariants. Turaev-Viro invariants are parameterized by a root of unity q, thus it is convenient to denote them by $TV(M)_q$, where M is a 3-manifold. Some of the interesting results of Chapter 1 are outlined in the following two theorems (see also [S1], [S3], [S4], [S5]).

THEOREM. (Remark 1.2, Theorem 1.5, Corollary 1.6) *Every Turaev-Viro invariant $TV(M)_q$ can be naturally split into three non-trivial invariants $TV_0(M)_q$, $TV_1(M)_q$, and $TV_2(M)_q$.*

Theorem. (Theorem 1.11, Corollary 1.12)

$$TV_N(M)_q = (-1)^N TV_N(M)_{-q}, \text{ where } N \in \{0,1,2\}$$

$$TV_0(M)_q + TV_2(M)_q = \frac{1}{2}(TV(M)_q + TV(M)_{-q}), \text{ and}$$

$$TV_1(M)_q = \frac{1}{2}(TV(M)_q - TV(M)_{-q}).$$

In Section 6 of Chapter 1 we solve the following conjecture (see also [S5], [S3]):

Conjecture. ([KL1]) Consider an arbitrary closed 3-manifold M, and let X be a special spine for M. Let n_e be the number of closed surfaces contained in X that have even Euler characteristic and n_o the number of closed surfaces in X that have odd Euler characteristic. Let $q = e^{2\pi i/8}$. Then (i) either $n_e = n_o$, or $n_o = 0$, and this alternative is invariant for every special spine of M; (ii) $n_o = 0 \Leftrightarrow TV(M)_q = TV(M^3)_{-q} \in \mathbf{Z}$.

As for any new invariant, it is important to know how strong it is. Usually, the first test to make is to check how well it distinguishes lens spaces. The next theorem from Section 7 of Chapter 1 gives a simple and complete answer to this question (see also [S2]). For any $v \in \mathbf{Z}$ define a function $h_v \colon \mathbf{Z} \to \mathbf{Z}_2$ by the formula

$$h_v(x) = \begin{cases} 1 & \text{if } x \equiv \pm 1 \bmod v \\ 0 & \text{if } x \not\equiv \pm 1 \bmod v \end{cases}$$

Theorem 1.27. *Two lens spaces $L(p_1, q_1)$ and $L(p_2, q_2)$ are not distinguished by Turaev-Viro invariants iff $p_1 = p_2 = p$ and for any divisor $v > 2$ of p we have $h_v(q_1) = h_v(q_2)$.*

Chapter 2 is probably the most interesting and useful part of this dissertation. It is based on the paper [PrS] written recently in collaboration with Józef Przytycki. Using surgery presentation for p-periodic 3-manifolds, developed by Józef Przytycki (see Theorem 2.1) we prove the following intriguing result:

Theorem 2.3. *If a closed orientable 3-manifold M admits an action of a cyclic group $\mathbf{Z}_p$ where p is an odd prime integer and the fixed point set of the action is S^1 then $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$.*

Note that this theorem provides a non-trivial criterion for 3-manifolds admitting the described action. The simplest examples of 3-manifolds with $H_1(M; \mathbf{Z}_p) = \mathbf{Z}_p$ are lens spaces $L_{pn,q}$ (or more generally, (pn/q) Dehn surgeries on knots in S^3).

Theorem 2.3 was first announced as a conjecture[1] and partially proven[2] in April of 1999 [S9]. Recently (November, 1999), Adam Sikora announced a proof of the theorem [Si]. In fact, using some classical but involved algebraic topology, he obtained more general results implying our theorem. The surgery presentation of periodic 3-manifolds developed in Section 1 of Chapter 2 allowed us to find an elementary proof of Theorem 2.3.

Theorem 2.3 is not true for $p = 2$, see Remark 2.20. An interesting criterion for 2-periodic rational homology spheres is provided by the following theorem.

THEOREM 2.4. *Let M be a rational homology 3-sphere such that the group $H_1(M; \mathbf{Z})$ does not have elements of order 16. If M admits an orientation preserving action of $\mathbf{Z}_2$ with the fixed point set being a circle then the canonical decomposition of the group $H_1(M; \mathbf{Z})$ has even number of terms $\mathbf{Z}_2$ and even number of terms $\mathbf{Z}_4$, and arbitrary number of terms $\mathbf{Z}_8$.*

Chapter 3 of the thesis studies how quantum invariants reflect periodicity of 3-manifolds. We consider Dijkgraaf-Witten invariants ([DW]) and the simplest known quantum invariants discovered by H. Murakami, T. Ohtsuki, and M. Okada ([MOO]). This part of our research was largely influenced by P. Gilmer's paper [G], where he obtained similar results for Reshetikhin-Turaev and Turaev-Viro invariants, and by ideas of J. Kania-Bartoszyńska and J. Przytycki in [KB-Pr].

Denote by $MOO^*_N(M)$ the Murakami-Ohtsuki-Okada invariants of order N (renormalized as in Section 1 of Chapter 3). We prove the following theorem:

THEOREM 3.6. *Let M be a closed oriented 3-manifold and $\mathbf{Z}_p$ acts on M with the fixed point set S^1 for some prime p. Let M_* be the orbit space of the action. If N is an odd integer coprime to p, then*

$$MOO^*_N(M) = \begin{cases} (MOO^*_N(M_*))^p \pmod{p} & \textit{if } N \equiv 1 \textit{ mod } 4, \\ \pm(MOO^*_N(M_*))^p \pmod{p} & \textit{if } N \equiv 3 \textit{ mod } 4. \end{cases}$$

Theorem 3.6 does not provide any restrictions for 3-manifolds to be p-periodic (see Remark 3.7). A non-trivial periodicity criterion is given by the following theorem.

THEOREM 3.8. *Suppose that a closed orientable 3-manifold M admits a $\mathbf{Z}_p$ action with the fixed point set equal S^1 for some odd prime p. Then either $MOO^*_p(M) = \pm 1$, or $MOO^*_p(M) = 0 \pmod{p}$.*

Let $DW_{G,\alpha}(M) \in \mathbf{Z}[\frac{1}{|G|}, q]$ be the Dijkgraaf-Witten invariant for a finite group G and $\alpha \in H^3(G, \mathbf{Q}/\mathbf{Z})$. We conclude Chapter 3 with the following result:

[1] The conjecture was obtained as a result of extensive computations performed with a program written in *Mathematica*.

[2] In the case when the orbit space of the action can be obtained from S^3 by an integer surgery on a knot.

THEOREM 3.13. *Let p be a prime number and M be an oriented 3-manifold such that $\mathbf{Z}_p$ acts on M. Let M_* be the orbit space of the action, that is $M_* = M/\mathbf{Z}_p$. For any finite group G whose order is not divisible by p and for any $\alpha \in H^3(G;\mathbf{Q}/\mathbf{Z})$ we have*
$$DW_{G,\alpha}(M) = (DW_{G,\alpha}(M_*))^p \pmod p.$$

In the last chapter, Chapter 4, we will study some simple skein modules and their relationship with quantum invariants.

In 1990 Lickorish showed how to construct Reshetikhin-Turaev ($SU(2)$) invariants from the Kauffman bracket skein module (see [Li2], [Li3], [Li4]). Recently, Y. Yokota constructed other quantum $SU(N)$-invariants using Lickorish method on HOMFLY skein module [Yo]. For an overview and some generalizations of Lickorish method see [B]. In Section 1 of Chapter 4 we define the simplest known skein modules, $\mathcal{S}$ and $\mathcal{S}_2$ (the later is also known as the *q-analog of the first homology*), and Lickorish's method for constructing 3-manifold invariants. In Section 2 of Chapter 4 we will show how to construct Murakami-Ohtsuki-Okada invariants ([MOO]) using Lickorish's method. In Section 3 of Chapter 4 we will show that if q is a root of unity then no other invariants can be obtained using Lickorish method on $\mathcal{S}$ and $\mathcal{S}_2$ (see Theorem 4.7).

Deloup's invariants $\tau(M, G, \xi)$ (see [De]), essentially introduced by his teacher V. Turaev via a simple example of modular categories [T2], are an interesting generalization of the Murakami-Ohtsuki-Okada invariants for 3-manifolds. Here G is a finite abelian group and $\xi : G \to \mathbf{Q}/\mathbf{Z}$ is a quadratic form. Deloup showed that these invariants depend only on the first Betti number and the linking form of the manifold. Recently, Deloup and Gille proved that these invariants keep all the information about the linking form and the first Betti number ([DG]). Moreover, they proved that all the information can, in fact, be recovered from two special cases: $\tau(M, \mathbf{Z}_{p^n}, (1/p^n))$, for prime p, and $\tau(M, \mathbf{Z}_{2^n} \times \mathbf{Z}_{2^n}, Q_n)$. Here $\tau(M, \mathbf{Z}_{p^n}, (1/p^n))$ coincides with the Murakami-Ohtsuki-Okada invariant of order p^n, and Q_n is the quadratic form defined by $Q_n(x, y) = xy/2^n$. As we mentioned above, Murakami-Ohtsuki-Okada invariants can be obtained using Lickorish construction on the skein module $\mathcal{S}_2$ (q-analogue of the first homology). Later in Chapter 4, we show that $\tau(M, \mathbf{Z}_{2^n} \times \mathbf{Z}_{2^n}, Q_n)$ can be similarly defined using another simple skein module, we call it *dichromatic skein module*, which turned out to be the tensor product of two copies of $\mathcal{S}_2$. The structure of the dichromatic skein module is studied in Section 5 of Chapter 4.

In Appendix A we revisit Dijkgraaf-Witten invariants and show how to define them using singular triangulations and simple 2-skeletons (a generalization of simple and special spines) of 3-manifolds. Other authors also studied Dijkgraaf-Witten invariants defined on special 2-polyhedra (see [Wak], [FG]), but their approach is valid only for the special 2-skeletons which are dual to triangulations.

In Appendix B we present tables with values of Turaev-Viro invariants for 239 closed oriented 3-manifolds of small complexity. There are many other tables of numeric values of Turaev-Viro and Reshetikhin-Turaev invariants (see [KL1], [KL2], [N]). An advantage of our tables is that the values of the invariants are presented not only as real numbers, but as elements of the corresponding cyclotomic field, i. e. as integer polynomials in q with every coefficient being an independent invariant.

In Appendix C we show how to determine linking pairing of a rational homology 3-sphere from its rational surgery presentation. It is well known that if M is a rational homology 3-sphere obtained by Dehn surgery on an (integer) framed oriented link L with a linking matrix A, then the linking pairing $\mathcal{L}_M$ can be presented by the matrix $-A^{-1}$ (see, for instance, [GL]). The well known proofs are based on consideration of the 4-manifold obtained obtained by attaching 2-handles to B^4 along $L \subset S^3 = \partial B^4$ according to the given framings. Thus, these proofs can not be naturally extended to rational surgery descriptions of 3-manifolds. In Appendix C we generalize this result to rational surgery presentations of M. Our proof is elementary and is based on a few simple facts from linear algebra.

CHAPTER 1
TURAEV–VIRO INVARIANTS OF 3-MANIFOLDS

Turaev-Viro invariants ([TV]) is a very important class of quantum invariants of 3-manifolds. First of all, they, unlike many other quantum invariants, can be calculated from a triangulation or from a simple spine of 3-manifolds. Secondary, Turaev ([T1], [T2]) and Walker ([W]) independently proved that Turaev-Viro invariants of orientable 3-manifolds coincide with the absolute value of Reshetikhin-Turaev invariants ([RT]), the first and the best known quantum invariants. In sections 1 and 2 we define Turaev-Viro invariants on simple spines of 3-manifolds. In section 3 we show that each Turaev-Viro invariant $TV(M)_q$ is a sum of three invariants $TV_0(M)_q, TV_1(M)_q$, and $TV_2(M)_q$. It follows from Turaev-Walker theorem that if q^2 is a primitive root of unity of an odd degree then, up to normalization, $TV_0(M)_q$ coincides with the square of the absolute value of the so–called $SO(3)$-invariant $\tau_e(M)$ defined in [T2]. For a connection between $SO(3)$-invariants and the Reshetikhin–Turaev invariants, see [KM] and [BHMV]. In section 4, using suitable normalizations we make the numbers $(TV_0(M)_q + TV_2(M)_q)$, $TV_0(M)_q$, and $TV_1(M)_q$ to be invariant under removing of 3-balls. That allows us to define these three invariants on a triangulation of a closed 3-manifold M. In section 5 we prove the following important formula:

$$TV_N(M)_q = (-1)^N TV_N(M)_{-q}, \text{ where } N \in \{0, 1, 2\},$$

which, in particular, implies

$$TV_0(M)_q + TV_2(M)_q = \frac{1}{2}(TV(M)_q + TV(M)_{-q}), \text{ and}$$

$$TV_1(M)_q = \frac{1}{2}(TV(M)_q - TV(M)_{-q}).$$

In section 6 we solve a conjecture of Kauffman and Lins formulated in [KL1] and give explicit formulas for Turaev-Viro invariants with the parameter $r = 4$. In section 7 we provide a simple criterion answering whether two lens spaces can be distinguished by Turaev-Viro invariants or not.

The results of this chapter where obtained during the author's work and study at the Chelyabinsk State University, Russia, under supervision of S. V. Matveev. The material from sections 1–6 was published in [S1], [S3], [S4], and [S5]. The material of section 7 was essentially published in [S2].

1.1 Simple polyhedra and their local moves

A 2-dimensional polyhedron X is called *simple* if the link of any point of X is homeomorphic to one of the following polyhedra: (1) a circle, (2) a circle with two radii, (3) a circle with three radii, (4) the segment $[0, 1]$, (5) a wedge of three segments with a common endpoint.

The set of points of a simple polyhedron X having links of types (4) or (5) is called the *boundary* of X and denoted by ∂X. The points with links of type (3) are called *vertices* of X. By an *edge* of X we mean a connected component of the set of points having the links of type (2).

Simple polyhedra are also called fake surfaces. This class of polyhedra generates the class of special polyhedra. Recall that a simple polyhedron X is called *special* if $\partial X = \emptyset$ and each 2-component of X is a 2-cell.

A simple polyhedron X with $\partial X = \emptyset$ is called a *simple spine* of a compact 3-manifold M with $\partial M \neq \emptyset$ if there exists an embedding $i \colon X \to M$ such that $M \searrow i(X)$, i.e. M collapses onto $i(X)$. In the case of a closed M, a polyhedron X is called a *simple spine* of M if it is a simple spine of M with an open 3-ball removed. A simple spine is called *special* if it is a special polyhedron. It is known that every compact connected 3-manifold has a special spine (see [Ca], [M1]).

Let us introduce four special polyhedra-with-boundary $P_1, \ldots, P_4$. Let P_1 be the polyhedron obtained from a disc D^2 by attaching two semidiscs along two parallel chords, h_1 and h_2 of D^2. The polyhedron P_2 is obtained from D^2 by attaching a semidisc along h_2 and the second one along a simple curve l in D^2 that has the same endpoints as h_1 and intersects h_2 transversally in exactly two points. Let $R = R_1 \cup R_2 \cup R_3$ be a triod consisting of three radii of the disc D^2. The polyhedron P_3 is obtained from the polyhedron $(D^2 \times \{0\}) \cup (R \times I)$ by attaching a semidisc along a chord $h_1 \subset D^2$ that intersects the radius R_1 in just one interior point. The polyhedron P_4 is obtained from $(D^2 \times \{0\}) \cup (R \times I)$ by attaching a semidisc along a simple curve that has the same endpoints as h_1 and intersects the triod R in exactly two points lying on on R_2 and R_3.

By an $\mathcal{L}$*-move* on simple polyhedra we mean a replacement of a fragment homeomorphic to P_1 by P_2. By an $\mathcal{M}$*-move* on simple polyhedra we mean a replacement of a fragment homeomorphic to P_3 by P_4 (for details, see [M1]).

Let a circle c bounds a 2-disc in a 2-component of a special polyhedron X. By $\mathcal{B}$*–move* we mean attaching an additional 2-disc to X along c (for details see [TV]).

It is proved in [M1] that any two special spines of a 3-manifold can be transformed one to the other by a sequence of the moves $\mathcal{M}^{\pm 1}$ and $\mathcal{L}^{\pm 1}$. Note that applying $\mathcal{L}$ several times, one can transform any simple spine into a special one. Thus, Matveev's theorem is true for simple spines as well.

A $\mathcal{B}$–move on a simple spine of a 3-manifold M corresponds to removing of one 3-ball from M.

1.2 The Turaev–Viro invariants

Throughout this chapter, let us fix $r \geq 3$ and a root of unity q of degree $2r$ such that q^2 is a primitive root of degree r.

In this section we recall how V. G. Turaev and O. Y. Viro define their invariants on a simple polyhedron X (cf. [TV]). Let $v_1, \dots, v_d$ be the vertices of X, let $e_1, \dots, e_f$ be the edges of ∂X and let $\Gamma_1, \dots, \Gamma_b$ be the 2-components of X.

By a *coloring* of X we mean an arbitrary mapping

$$\varphi\colon \{\Gamma_1, \dots, \Gamma_b\} \to \mathbf{Z}_{r-1} = \{0, 1, \dots, r-2\}.$$

A triple $(i, j, k) \in \mathbf{Z}_{r-1}^3$ will be called *admissible* if

$$2r - 4 \geq i + j + k \equiv 0 \pmod 2,$$

$$|i - j| \leq k \leq i + j.$$

A coloring φ is called *admissible* if for any edge E of $X - \partial X$ the colors of the 2-components incident to E form an admissible triple. Let us denote the set of admissible triples by *adm* and the set of admissible colorings of X by $Adm(X)$.

By a *coloring* of a regular graph G we shall mean any mapping of the set of its edges to $\mathbf{Z}_{r-1}$. Let us denote the set of the colorings of X by $Col(X)$. Any coloring φ of a simple polyhedron X induces in a natural way a coloring $\partial\varphi$ of its boundary ∂X: an edge of ∂X takes the color of the 2-component of X in whose boundary this edge is contained.

Let $\Gamma_i, \Gamma_j, \Gamma_k$ be the 2-components incident to an edge E of X and let $\varphi \in Adm(X)$. We shall say that the unordered triple $\{\varphi(\Gamma_i), \varphi(\Gamma_j), \varphi(\Gamma_k)\}$ is a *color of the edge E*. There are six *wings* incident to any vertex v of a simple polyhedron. Suppose they receive under φ the values $i, j, k, l, m, n \in \mathbf{Z}_{r-1}$. A 6-tuple $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ is called a *color of the vertex v* if $\{i, j, k\}$ is a color of some edge incident to v and (i, l), (j, m), (k, n) are the colors of opposite 2-components incident to v.

For an integer $n > 0$ set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q.$$

Set also $[0]_q = [0]_q! = 1$. For a color $\{i, j, k\}$ of an edge set

$$\Delta_q(i, j, k) = \left(\frac{[\underline{i} + \underline{j} - \underline{k}]_q! [\underline{i} + \underline{k} - \underline{j}]_q! [\underline{j} + \underline{k} - \underline{i}]_q!}{[\underline{i} + \underline{j} + \underline{k} + 1]_q!} \right)^{1/2}$$

where $\underline{i} = i/2$. Note that the expression in the round brackets presents a real number. By the square root $x^{1/2}$ of a real number x we mean the positive root of $|x|$ multiplied by $\sqrt{-1}$ if $x < 0$.

Let $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ be a color of some vertex v. The *symbol* of v is defined by the following formula

$$|T_v^\varphi|_q = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_q = (\sqrt{-1})^{-(i+j+k+l+m+n)} \Delta_q(i,j,k)\Delta_q(i,m,n)\times$$

$$\times \Delta_q(j,l,n)\Delta_q(k,l,m) \begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix}_q,$$

where

$$\begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix}_q = \sum_z (-1)^z [z+1]_q! \{[z - \underline{i} - \underline{j} - \underline{k}]_q! [z - \underline{i} - \underline{m} - \underline{n}]_q! [z - \underline{j} - \underline{l} - \underline{n}]_q! \times$$

$$\times [z - \underline{k} - \underline{l} - \underline{m}]_q! [\underline{i} + \underline{j} + \underline{l} + \underline{m} - z]_q! [\underline{i} + \underline{k} + \underline{l} + \underline{n} - z]_q! [\underline{j} + \underline{k} + \underline{m} + \underline{n} - z]_q! \}^{-1}.$$

Here z runs through the non-negative integers such that all expressions in the square brackets are non-negative. For $i \in \mathbf{Z}_{r-1}$ put

$$w_{i,q} = (\sqrt{-1})^i [i+1]_q^{1/2}.$$

For $\varphi \in Adm(X)$ put

$$|X, \varphi|_q = \prod_{i=1}^{b} w_{\varphi(\Gamma_i),q}^{2\chi(\Gamma_i)} \prod_{s=1}^{f} w_{\partial\varphi(e_s),q}^{\chi(e_s)} \prod_{j=1}^{d} |T_{v_j}^\varphi|_q,$$

where χ is the Euler characteristic (the 2-components of X and the edges of ∂X are thought to be open, so if e_s is homeomorphic to $\mathbf{R}$ then $\chi(e_s) = -1$ and if e_s is homeomorphic to S^1 then $\chi(e_s) = 0$).

The Turaev-Viro invariant for the simple polyhedron X is given by

$$TV(X)_q = \sum_{\varphi \in Adm(X)} |X, \varphi|_q.$$

It is proved in [TV] that $TV(X)_q$ is invariant under the moves $\mathcal{L}^{\pm 1}$ and $\mathcal{M}^{\pm 1}$. It follows from Matveev's theorem that if X is a simple spine of a 3-manifold M then $TV(M)_q = TV(X)_q$ is a topological invariant of M.

Note that in [TV] a different normalization is used. The original Turaev-Viro invariant is given by the formula

$$TV^*(X)_q = \omega^{-2\chi(X)+\chi(\partial X)} TV(X)_q,$$

where $\omega = \sqrt{2r}/|q - q^{-1}|$. It is proved in [TV] that $TV^*(X)_q$ is invariant under $\mathcal{B}^{\pm 1}$ also.

REMARK 1.1. It is easily seen that if q is a primitive root of unity of degree $2r$ and $\partial X = \emptyset$ then the numbers $|X, \varphi|_q$, and therefore the numbers $TV(X)_q$ and $TV^*(X)_q$, lie in $\mathbf{Q}(q)$.

1.3 The summand–invariants

The set of 2-components of X that receive odd colors under a coloring $\varphi \in Adm(X)$ forms a surface embedded in X. We denote this surface by $S(\varphi)$. Note that $\partial S(\varphi) \subseteq \partial X$.

Present the set $Adm(X)$ as a disjoint union of subsets $Adm_0(X), Adm_1(X)$ and $Adm_2(X)$, where

0) $\varphi \in Adm_0(X) \Leftrightarrow (\varphi \in Adm(X))$ & $(S(\varphi) = \emptyset)$;

1) $\varphi \in Adm_1(X) \Leftrightarrow (\varphi \in Adm(X))$ & $(\chi(S(\varphi)) \equiv 1 \pmod 2)$;

2) $\varphi \in Adm_2(X) \Leftrightarrow (\varphi \in Adm(X))$ & $(S(\varphi) \neq \emptyset)$ & $(\chi(S(\varphi)) \equiv 0 \pmod 2)$.

For any coloring α of ∂X and $N \in \{0, 1, 2\}$ put

$$\Omega_N(X, \alpha)_q = \sum_{\substack{\varphi \in Adm_N(X) \\ \partial\varphi = \alpha}} |X, \varphi|_q.$$

If $\{\varphi \in Adm_N(X) \colon \partial\varphi = \alpha\} = \emptyset$, then $\Omega_N(X, \alpha)_q = 0$. Put also

$$TV_N(X)_q = \sum_{\alpha \in Col(\partial X)} \Omega_N(X, \alpha)_q,$$

where the sum is taken over all colorings of ∂X.

REMARK 1.2. $TV(X)_q = TV_0(X)_q + TV_1(X)_q + TV_2(X)_q$.

REMARK 1.3. If q is a primitive root of unity of degree $2r$ then for every simple polyhedron X with $\partial X = \emptyset$ we have $TV_N(X)_q \in \mathbf{Q}(q)$, for any $N \in \{0, 1, 2\}$ (see Remark 1.1).

LEMMA 1.4. *Let a simple polyhedron X be the union of simple polyhedra Y and Z and let each connected component of $T = Y \cap Z$ be a connected component of both ∂Y and ∂Z. Then for any coloring β of ∂X we have*

$$\Omega_0(X, \beta)_q = \sum_{\alpha \in Col(T)} \Omega_0(Y, \alpha \cup (\beta|_{Y \cap \partial X}))_q \, \Omega_0(Z, \alpha \cup (\beta|_{Z \cap \partial X}))_q,$$

$$\Omega_1(X,\beta)_q = \sum_{\substack{\alpha\in Col(T)\\ K+L\equiv 1(2)}} \Omega_K(Y,\alpha\cup(\beta|_{Y\cap\partial X}))_q\,\Omega_L(Z,\alpha\cup(\beta|_{Z\cap\partial X}))_q,$$

$$\Omega_2(X,\beta)_q = \sum_{\substack{\alpha\in Col(T)\\ K+L=2\ or\ 4}} \Omega_K(Y,\alpha\cup(\beta|_{Y\cap\partial X}))_q\,\Omega_L(Z,\alpha\cup(\beta|_{Z\cap\partial X}))_q.$$

PROOF. This follows from the equalities

$$|X,\varphi|_q = |Y,(\varphi|_Y)|_q\cdot|Z,(\varphi|_Z)|_q,$$

where $\varphi\in Adm(X)$ (see Lemma 4.2.A in [TV]), and $\chi(X)=\chi(Y)+\chi(Z)$. □

THEOREM 1.5. *Let X be a simple 2-polyhedron and α be a coloring of ∂X. Then for every $N\in\{0,1,2\}$ the number $\Omega_N(X,\alpha)_q$ is invariant under $\mathcal{L}^{\pm 1}$ and $\mathcal{M}^{\pm 1}$.*

PROOF. Let us show that the number $\Omega_N(X,\alpha)_q$ is invariant under $\mathcal{L}$. The case of the $\mathcal{M}$-move is similar. By Lemma 1.4, it is sufficient to prove that $\Omega_N(P_1,\gamma)_q = \Omega_N(P_2,\gamma)_q$ for any $N\in\{0,1,2\}$, where P_1 and P_2 are the polyhedra from the definition of the $\mathcal{L}$-move and γ is a coloring of the graph $\partial P_1=\partial P_2$. It is easy to check that for any γ there is a unique $K\in\{0,1,2\}$ such that $\{\varphi\in Adm(P_i)\colon\partial\varphi=\gamma\}\subset Adm_K(P_i)$, for $i=1,2$. Therefore

$$\Omega_N(P_1,\gamma)_q = \sum_{\substack{\varphi\in Adm(P_1)\\ \partial\varphi=\gamma}} |P_1,\varphi|_q \quad\text{and}\quad \Omega_N(P_2,\gamma)_q = \sum_{\substack{\psi\in Adm(P_2)\\ \partial\psi=\gamma}} |P_2,\psi|_q$$

if $N=K$, and $\Omega_N(P_1,\gamma)_q=\Omega_N(P_2,\gamma)_q=0$ if $N\neq K$. It is proved in Lemma 4.4.A of [TV] that the sums are equal. □

COROLLARY 1.6. *Let X be a simple spine of a 3-manifold M. Then $TV_N(M)_q = TV_N(X)_q$ is an invariant of M for every $N\in\{0,1,2\}$.*

1.4 The summand-invariants and a triangulation

The summand invariants are not invariant under the $\mathcal{B}$–move. This prevents us from defining these invariants on triangulations of 3-manifolds. Here we modify the invariants TV_0, TV_1, and TV_0+TV_2 to make them invariant under removing of 3-balls.

Put $\omega_0=\sqrt{r}/|q-q^{-1}|$ and $\omega=\sqrt{2r}/|q-q^{-1}|$. Let X be a simple polyhedron. Put

$$\Omega_0^*(X,\alpha)_q = \omega_0^{-2\chi(X)+\chi(\partial X)}\Omega_0(X,\alpha)_q,$$
$$\Omega_1^*(X,\alpha)_q = \omega^{-2\chi(X)+\chi(\partial X)}\Omega_1(X,\alpha)_q,$$

and

$$\Omega_e^*(X,\alpha)_q = \omega^{-2\chi(X)+\chi(\partial X)}(\Omega_0(X,\alpha)_q+\Omega_2(X,\alpha)_q).$$

Lemma 1.7. *Let X be a simple 2-polyhedron and α be a coloring of ∂X. Then the numbers $\Omega_0^*(X,\alpha)_q$, $\Omega_1^*(X,\alpha)_q$, and $\Omega_e^*(X,\alpha)_q$ are invariant under $\mathcal{B}$.*

Proof. It follows immediately from the definition of the number $|X,\varphi|_q$ that the number $\Omega_0^*(X,\alpha)_q$ is invariant under $\mathcal{B}$ if

$$\omega_0^2 = w_j^{-2} \sum_{\substack{k,l\equiv 0(2)\\ k,l:\{j,k,l\}\in adm}} w_k^2 w_l^2$$

for any even $j \in \mathbf{Z}_{r-1}$, and $\Omega_1^*(X,\alpha)_q$, $\Omega_e^*(X,\alpha)_q$ are invariant under $\mathcal{B}$ if

$$\omega^2 = w_j^{-2} \sum_{k,l:\{j,k,l\}\in adm} w_k^2 w_l^2$$

for any $j \in \mathbf{Z}_{r-1}$.

The second equality is proved in [TV]. The proof of the first one is similar. First of all, let us check that

$$w_j^{-2} \sum_{\substack{k,l\equiv 0(2)\\ k,l:\{j,k,l\}\in adm}} w_k^2 w_l^2 = w_0^{-2} \sum_{\substack{s\equiv 0(2)\\ 0\le s\le r-2}} w_s^4 \tag{1.1}$$

for every even number $j \in \mathbf{Z}_{r-1}$.

Let T be a polyhedron obtained from a disc D^2 by attaching one semidisc along a diameter of D^2. The polyhedron T consists of three 2-cells $\Gamma_1, \Gamma_2, \Gamma_3$. Let a polyhedron T_i is obtained from T by attaching a 2-disc along a circle that belongs to the 2-cell Γ_i, where $i = 1$ or 2.

For any $j \in \mathbf{Z}_{r-1}$ we define a coloring β of ∂T_1 and ∂T_2 as follows: $\beta(\Gamma_1) = \beta(\Gamma_3) = j$, $\beta(\Gamma_2) = 0$.

By definition, we have

$$\Omega_0(T_1,\beta)_q = w_0^2 w_j^2 \sum_{\substack{k,l\equiv 0(2)\\ k,l:\{j,k,l\}\in adm}} w_k^2 w_l^2,$$

$$\Omega_0(T_2,\beta)_q = w_j^4 \sum_{\substack{s\equiv 0(2)\\ 0\le s\le r-2}} w_s^4.$$

Note that T_1 and T_2 are connected by $\mathcal{L}$-move, therefore $\Omega_0(T_1,\alpha)_q = \Omega_0(T_2,\alpha)_q$. This gives us the equality (1.1).

Clearly, $w_0 = 1$. Thus we have to prove that

$$\sum_{t=0}^{[r/2]-1} w_{2t}^4 = -r/(q-q^{-1})^2.$$

The proof of this equality is straightforward. □

Corollary 1.8. *Let X be a simple spine of a 3-manifold M. Then the numbers $TV_0^*(M)_q = \Omega_0^*(X)_q$, $TV_1^*(M)_q = \Omega_1^*(X)_q$, and $TV_e^*(M) = \Omega_e^*(X)_q$ are invariants of M under removing of 3-balls.*

Now we can define the invariants TV_0^*, TV_1^*, and TV_e^* on a triangulation of a 3-manifold M just like the Turaev–Viro invariants were defined in [TV]. For simplicity we will restrict ourselves to the case of closed 3-manifolds only.

Let M be a closed triangulated 3-manifold. Let a be the number of vertices of M, let $E_1, \dots, E_b$ be the edges of M, and let $T_1, \dots, T_d$ be the 3-simplexes of M. By a *coloring* of M we mean an arbitrary map $\varphi\colon \{E_1, \dots, E_b\} \to \mathbf{Z}_{r-1}$. A coloring φ of M is called *admissible* if for any 2-symplex A of M the colors of the three edges of A form an admissible triple. Denote the set of admissible colorings of M by $Adm(M)$. We will denote by $Adm_0(M)$ the set of admissible colorings of M by even numbers, and by $Adm_e(M)$ the set of admissible colorings of M such that

$$v - t + f \equiv 0 \pmod 2,$$

where v is the number of 3-simplexes containing an edge colored by an odd number, t is the number of 2-simplexes containing an edge colored by an odd number, and f is the number of edges colored by odd numbers. Note that $Adm_0(M) \subset Adm_e(M)$. Set $Adm_1(M) = Adm(M) - Adm_e(M)$. A 6-tuple $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ is called a *color* of a 3-simplex T_s if i, j, k are the colors of the edges of some 2-face of T_s and (i, l), (j, m), (k, n) are the pairs of colors of opposite edges of T_s. Let

$$|T_s^\varphi|_q = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_q .$$

For $\varphi \in Adm(M)$ put

$$|M, \varphi|_q = \prod_{i=1}^{b} w^2_{\varphi(E_i),q} \prod_{s=1}^{d} |T_s^\varphi|_q.$$

Proposition 1.9. *For any closed triangulated 3-manifold M we have*

$$TV_0^*(M)_q = \omega_0^{-2a} \sum_{\varphi \in Adm_0(M)} |M, \varphi|_q,$$

$$TV_1^*(M)_q = \omega^{-2a} \sum_{\varphi \in Adm_1(M)} |M, \varphi|_q,$$

$$TV_e^*(M)_q = \omega^{-2a} \sum_{\varphi \in Adm_e(M)} |M, \varphi|_q.$$

Proof. Let X be the union of the closed barycentric stars of the edges of M. It is obvious that X is a special polyhedron. By a finite number of $\mathcal{M}^{\pm 1}, \mathcal{L}^{\pm 1}$ and $\mathcal{B}^{-1}$ moves on X we get a simple spine of M. Each coloring φ of M induces a dual coloring φ^* of X, and it is easy to check that $|M, \varphi|_q = |X, \varphi^*|_q$ and $\chi(X) = a$, which establishes the formulas. $\square$

1.5 Values of the summand-invariants

In this section we will express the numbers $TV_0(M)_q + TV_2(M)_q$ and $TV_1(M)_q$ via the Turaev–Viro invariants.

Let X be a special polyhedron. Fix a number $r \geq 3$ and a coloring $\varphi \in Adm(X)$. A vertex v of the colored polyhedron X is called a *switch-vertex* if the sum of all odd numbers in the color of v is congruent to 2 modulo 4. The following important lemma was proven with help of Artem Macovetski.

Lemma 1.10. *Let X be a special polyhedron. Then for any $\varphi \in Adm(X)$ we have*

$$|X, \varphi|_q = (-1)^{\chi(S(\varphi))+x} |X, \varphi|_{-q},$$

where x is the number of the switch-vertices of X.

Proof. It is easy to see that

$$[n]_q = (-1)^{n-1}[n]_{-q},$$

$$[n]_q! = (-1)^{n(n-1)/2}[n]_{-q}!,$$

$$w_{i,q}^2 = (-1)^i w_{i,-q}^2,$$

$$\Delta_q^2(i,j,k) = \begin{cases} \Delta_{-q}^2(i,j,k), & \text{if } i,j,k \text{ are even,} \\ -\Delta_{-q}^2(i,j,k), & \text{otherwise.} \end{cases}$$

Let $\begin{pmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{pmatrix}$ be the color of a vertex v of X under φ. Then we have

$$\begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_q = (-1)^{\frac{1}{2}\sum_{\substack{s,t=1 \\ s \leq t}}^{6} i_s i_t} \begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_{-q}. \tag{1.2}$$

There are three possibilities for the color of v.

1) Each number in the color of v is even (*even vertex*). Then the sign in (1.2) is plus.

2) There are four odd numbers in the color of v (*fourfold vertex*). Let i_1, i_2, i_4, i_5

be the odd numbers, then from (1.2) we have

$$\begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_q = (-1)^{\frac{i_1+i_2+i_4+i_5}{2}+1} \begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_{-q}.$$

Hence if v is a switch-vertex, then the sign in (1.2) is plus, otherwise minus.

3) There are three odd and three even numbers in the color of v (*threefold vertex*). Let i_1, i_2, i_3 be the even numbers, then from (1.2) we have

$$\begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_q = (-1)^{\frac{i_1+i_2+i_3}{2}+1} \begin{bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{bmatrix}_{-q}.$$

By the *cost* of an edge E we mean the half-sum of the numbers from the color of E. Let E' be a half-edge of an edge E. By the *cost* of the half-edge E' we mean the cost of E. We call an edge or a half-edge *bad* if its color is *even* (that is all three numbers in the color of the edge are even) and its cost is even. Let us show that the number of threefold vertices with a bad half-edge is even. It is sufficient to prove that the number of the bad half-edges incident to an even vertex is even, but this statement follows from the fact that the sum of all 4 costs of the half-edges incident to an even vertex is even. Hence, we can think that for any threefold vertex the sign in (1.2) is plus.

Let us denote the number of the odd-colored edges of X by e, the number of the threefold vertices by n_3, and the number of the fourfold vertices by n_4. Denote the number of the odd colored 2-components of X by c. Then we have $c = \chi(S(\varphi)) - n_3 - n_4 + e$. Hence

$$|X, \varphi|_q = (-1)^{c+n_4-x+e}|X, \varphi|_{-q} = (-1)^{\chi(S(\varphi))-n_3-x}|X, \varphi|_{-q}.$$

It is easy to see that for any admissible coloring φ of a special polyhedron X the number n_3 is even. This finishes the proof. □

Let SX be the set of the singular points of X. Note that SX is a regular graph of degree 4. Denote by V the set of vertices of X, by $N(V, SX)$ a closed regular neighborhood of V in SX, and by $N(V, X)$ a closed regular neighborhood of V in X. The intersection of the union of the open edges with each connected component of $N(V, SX)$ consists of 4 half-open 1-cells, which are called *thorns*. The intersection of the union of the open 2-cells with each connected component of $N(V, X)$ consists of six half-open 2-cells, which are called *wings*.

Let v be a vertex of X, and let $N(v, M)$ be a closed regular neighborhood of v. Chose a thorn t in $N(v, M)$ and a small normal disc D for it. Any orientation α of $N(v, M)$ induces an orientation $\alpha|_D$ of D according to the following convention: $\alpha|_D$ together with the outward orientation of t should give the orientation α. Note that $\alpha|_D$ induces a cyclic order on the set of wings adjacent to t.

Regular neighborhood $N(V, M)$ consists of 3-balls $N(v, M)$, $v \in V$. Choose orientations for the 3-balls. Let E be an edge of X. It contains two thorns t_1, t_2. Let $W_1^{(i)}, W_2^{(i)}, W_3^{(i)}$ be the wings adjacent to t_i, where $i = 1, 2$. As above, the orientation of $N(V, M)$ induces a cyclic order on the set

$$\{W_1^{(i)}, W_2^{(i)}, W_3^{(i)}\}, \qquad for \quad i = 1, 2.$$

The 2-cells of X determine the natural bijection

$$f: \{W_1^{(1)}, W_2^{(1)}, W_3^{(1)}\} \to \{W_1^{(2)}, W_2^{(2)}, W_3^{(2)}\}.$$

We shall say that the edge E is *odd* if the bijection f preserves the cyclic order on the wings, and *even* otherwise.

THEOREM 1.11. *Let X be a special spine of a 3-manifold M. Then for any $\varphi \in Adm(X)$ we have*

$$|X, \varphi|_q = (-1)^{\chi(S(\varphi))} |X, \varphi|_{-q}.$$

PROOF. Let x be the number of switch-vertices of the pair (X, φ). By Lemma 1.10, it is sufficient to prove that this number is even.

Consider the coloring $\overline{\varphi}: \{\Gamma_1, \ldots, \Gamma_b\} \to \mathbf{Z}_4$ such that

$$\overline{\varphi}(\Gamma_i) = \begin{cases} 0, & \text{if } \varphi(\Gamma_i) \equiv 0 \pmod 2, \\ 1, & \text{if } \varphi(\Gamma_i) \equiv 1 \pmod 4, \\ 3, & \text{if } \varphi(\Gamma_i) \equiv 3 \pmod 4, \end{cases}$$

for any $1 \leq i \leq b$.

Fix an orientation of $N(V, M)$. Then each edge of SX becomes odd or even. Let G be the union of the edges of X with colors $\{0, 1, 3\}$ under the coloring $\overline{\varphi}$. Let $\Omega_1, \ldots, \Omega_p$ be the middle points of the odd edges of G. Consider a graph G'. The set of vertices of G' consists of the vertices of G and of the points $\Omega_1, \ldots, \Omega_p$. The set of edges of G' consists of the even edges of G and of the halves of the odd edges of G. Thus, each odd edge of G gives 2 edges in G'. The orientation of $N(V, M)$ and the coloring $\overline{\varphi}$ give the orientation of the graph G'. Let $v_1, \ldots, v_t$ be the vertices of G'. We will denote by a_i the number of incoming and by b_i the number of outgoing edges for the vertex v_i. We have $(a_i - b_i) \equiv 2 \pmod 4$ iff v_i is either the switch-vertex or the middle point of an odd edge, and $(a_i - b_i) \equiv 0 \pmod 4$ otherwise. The number of vertices with the condition $(a_i - b_i) \equiv 2 \pmod 4$ is even for any oriented graph, because $\sum_{i=1}^{t} (a_i - b_i) = 0$.

It remains to prove that the number p is even. Let θ be the number of odd edges of X with the color $\{0, 1, 1\}$ under the coloring $\overline{\varphi}$. Then 1-colored (by $\overline{\varphi}$) 2-cells pass $(2\theta + p)$ times along the odd edges of X. Note that each 2-component of X passes along the odd edges of X even number of times (this is true for every special spine;

see, for instance, [Fo]). Therefore the number $(2\theta + p)$ is even and p is even. $\square$

COROLLARY 1.12. *For any 3-manifold M and any q we have*

$$TV_N(M)_q = (-1)^N TV_N(M)_{-q}, \quad where \quad N \in \{0, 1, 2\},$$

$$TV_0(M)_q + TV_2(M)_q = \frac{1}{2}(TV(M)_q + TV(M)_{-q}),$$

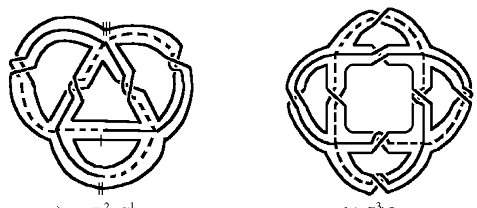
$$TV_1(M)_q = \frac{1}{2}(TV(M)_q - TV(M)_{-q}).$$

REMARK 1.13. In the papers [S3], [S4], and [S5] we used the parameter $-\overline{q}$ instead of $-q$, but it is easy to see that $[n]_q = [n]_{\overline{q}}$.

1.6 Kauffman-Lins conjecture. The case r=4.

In [KL1] L. H. Kauffman and S. Lins put forward the following conjecture.

CONJECTURE. Consider an arbitrary closed 3-manifold M, and let X be a special spine for M. Let n_e be the number of closed surfaces contained in X that have even Euler characteristic and n_o the number of closed surfaces in X that have odd Euler characteristic. Let $q = e^{2\pi i/8}$. Then (i) either $n_e = n_o$, or $n_o = 0$, and this alternative is invariant for every special spine of M; (ii) $n_o = 0 \Leftrightarrow TV(M)_q = TV(M)_{-q} \in \mathbf{Z}$.



a) $\mathbf{R}P^2 \times S^1$ b) S^3/Q_{16}

Fig. 1.1

Note that if M is an orientable manifold, then part (i) of the conjecture follows from 8.3-8.4 in [TV]. If M is a non-orientable manifold, then there exists a counter example offered by S. V. Matveev: the manifold $\mathbf{R}P^2 \times S^1$. A neighborhood of the 1-skeleton of a special spine for $\mathbf{R}P^2 \times S^1$ is shown on Fig. 1.1a. For this spine we have: $n_o = 1$ and $n_e = 3$. This disproves the part (i) of the conjecture. (Let us give some explanation of Fig. 1.1a. The manifold $(\mathbf{R}P^2 \times S^1) - B^3$ collapses onto $m \times S^1 \cup_{\partial m} D^2$, where m is a Möbius band and D^2 is a disc attached along ∂m. The manifold $m \times S^1$ collapses onto a torus T^2, so $(\mathbf{R}P^2 \times S^1) - B^3$ collapses onto T^2 with a disc D^2, attached to it such that ∂D^2 winds twice around the longitude of T^2. There is an isotopy of ∂D^2 in T^2 taking our polyhedron to a special one with a neighborhood of the 1-skeleton pictured on Fig. 1.1a. The union of the 2-components marked by I, II, III is the torus T^2, the unmarked curve is the boundary of the disc D^2.)

Let us consider the manifold S^3/Q_{16} (A neighborhood of the 1-skeleton of a special spine for S^3/Q_{16} is shown on Fig. 1.1b. Explanations can be found in [M2]). We can see that $TV_q(S^3/Q_{16}) = TV_{-q}(S^3/Q_{16}) = 6$ but $n_o \neq 0$ for this manifold. So the implication $\Leftarrow$ of the part (ii) in the conjecture is not true. But if $n_o = 0$ then indeed $TV_q(M) = TV_{-q}(M) \in \mathbf{Z}$, as follows from the next lemma. Moreover, if $n_o = 0$ then the equation $TV_q(M) = TV_{-q}(M)$ is true for any parameter q, see Theorem 1.11 and Corollary 1.12.

Let us denote by $S_i(\varphi)$ the union of the 2-components $\Gamma_j \subset X$ such that $\varphi(\Gamma_j) = i$.

LEMMA 1.14. *Let X be a special spine of a compact 3-manifold M^3, let $r = 4$, and $\varphi \in AdmX$. Then*

$$|X, \varphi|_q = (-\sqrt{2})^{\chi(S_1(\varphi))}(-1)^n,$$

$$|X, \varphi|_{-q} = (\ \sqrt{2})^{\chi(S_1(\varphi))}(-1)^n,$$

where n is the number of vertices in the graph $G = \partial S_2(\varphi)$.

PROOF. The values of the symbols for vertices and the weights for 2-cells of the spine X in the case $r = 4$ are given in [KL1]. Their products give the above equalities. $\square$

1.7 Which lens spaces are distinguished by Turaev-Viro invariants?

In this section we will answer the following question: Given two lens spaces, do the Turaev-Viro invariants distinguish them? It follows from the definition of Turaev-Viro invariants that if q_1 and q_2 are roots of unity of the same order then $TV(M)_{q_2}$ can be obtained from $TV(M)_{q_1}$ using the Galois automorphism sending q_1 to q_2. Thus, in order to answer the above question it is sufficient to consider only Turaev-Viro invariants parameterized by $q = e^{\pi i/r}, r \geq 3$. In this case it will be convenient to use the notation $TV_r(M)$ instead of $TV(M)_q$. We say that two lens spaces are distinguished by Turaev-Viro invariants if there exists an $r > 2$ such that the values of TV_r on these lens spaces are different.

In [Ya] Yamada constructed examples of an arbitrary large number of homotopy equivalent (but not homeomorphic) lens spaces that are not distinguished by Reshetikhin-Turaev invariants RT_r. These spaces are not distinguished by Turaev-Viro invariants also because (see [T1], [T2], [W]) $|RT_r(M)|^2 = TV_r(M)$, where M is a closed orientable 3-manifold. However, this does not answer the question stated in the title of this section.

The main result of [Ya] is the following explicit formula (1.3), which gives the value of the Turaev-Viro invariants of the lens spaces (this formula was obtained from Jeffrey's formula for the Reshetikhin-Turaev invariants [Je]):

$$(1.3)\quad TV_r(L_{p,q}) = \begin{cases} \frac{1}{r}(1-\cos\frac{2\pi p^*}{r}), & \text{if } d=1;\\ \frac{2}{r}(1-\cos\frac{\pi p'^*}{r}), & \text{if } d=2,\ c \text{ is even};\\ \frac{d}{2r}, & \text{if either } d>2,\ c \text{ is even},\ q\equiv \pm 1 \pmod d,\\ & \text{or } d>2,\ c \text{ is odd},\ q\equiv \frac{d}{2}\pm 1 \pmod d;\\ 0, & \text{in all other cases.}\end{cases}$$

Here $d = g.c.d.(p, 2r)$, $c = \frac{2pr}{d^2}$, $p' = \frac{p}{2}$, and p^*, p'^* are integers such that $pp^* \equiv 1 \pmod r$, $\quad p'p'^* \equiv 1 \pmod r$, and $\frac{p'p'^*-1}{r}p'^*$ is even.

Below we use the following statement.

COROLLARY 1.15. ([Ya]) *If either $d = gcd(p, 2r) \le 2$ or $2pr/d^2$ and $d/2$ are odd integers, then $TV_r(L_{p,q})$ does not depend on q.*

REMARK 1.16. In [Ya] and [Je] the Reshetikhin-Turaev invariants are called *Chern-Simons-Witten invariants* and the Turaev-Viro invariants are called *absolute values of the Chern-Simons-Witten invariants.*

REMARK 1.17. Yamada defines p'^* as an integer such that we have $p'p'^* \equiv 1 \pmod{rq}$ and the number $(p'p'^* - 1)p'^*/(rq)$ is even; so p'^* depends on q. However (see Corollary 1.15), if $d = 2$, then $TV_r(L_{p,q})$ does not depend on the parameter q. Thus, in the case $d = 2$ we can suppose that $q = 1$.

There is a useful particular case of formula (1.3):

COROLLARY 1.18.

$$TV_3(L_{p,q}) = \begin{cases} 1/2, & \textit{if } p \textit{ is odd};\\ 1, & \textit{if } 4|p;\\ 0, & \textit{if } p \textit{ is even and } 4 \not| p.\end{cases}$$

PROOF. Let $p = 2^s 3^t u$ where u is coprime with 2 and 3. We need to consider the following 6 cases.

1) $\underline{s = 0,\quad t = 0}$ $(\Rightarrow d = 1)$. Since $pp^* \equiv 1 \pmod 3$, we have $3 \not| p^*$, and $\cos\frac{2\pi p^*}{3} = -1$. Therefore, by (1.3), $TV_3(L_{p,q}) = \frac{1}{2}$.

2) $\underline{s = 0,\quad t > 0}$ $(\Rightarrow d = 3,\quad q \equiv \pm 1 \pmod 3)$. By (1.3), $TV_3(L_{p,q}) = \frac{1}{2}$.

3) $\underline{s = 1,\quad t = 0}$ $(\Rightarrow d = 2, c \text{ is odd})$. By (1.3), $TV_3(L_{p,q}) = 0$.

4) $\underline{s = 1,\quad t > 0}$ $(\Rightarrow d = 6, c \text{ is odd})$. q is odd, therefore $q \not\equiv 3 \pm 1 \pmod 6$. By (1.3), $TV_3(L_{p,q}) = 0$.

5) $\underline{s > 1,\quad t = 0}$ $(\Rightarrow d = 2, c \text{ is even})$. From the conditions on p'^* we can conclude that $p'^* \equiv 3 \pm 1 \pmod 6$, i. e. $\cos\frac{\pi p'^*}{3} = -\frac{1}{2}$. By (1.3), $TV_3(L_{p,q}) = 1$.

6) $\underline{s > 1,\quad t > 1}$ $(\Rightarrow d = 6, c \text{ is even})$. Since $(q, p) = 1$, we have $q \equiv \pm 1 \pmod 6$. By (1.3), $TV_3(L_{p,q}) = 1$. $\square$

PROPOSITION 1.19. *If $p_1 \ne p_2$ then the lens spaces L_{p_1,q_1} and L_{p_2,q_2} are distinguished by Turaev-Viro invariants.*

Proof. Let R be an integer coprime to p_1 and p_2 such that $p_1 \not\equiv \pm p_2 \pmod R$, for instance, $R = 3p_1p_2 + 1$. Consider several cases.

1)Both p_1 and p_2 are odd. Take $r = R$. Then d_i =g.c.d.$(p_i, 2R) = 1, \quad i = 1, 2$. In this case, by (1.3), the two lens spaces are not distinguishable by Turaev-Viro invariants iff $\cos\frac{2\pi p_1^*}{r} = \cos\frac{2\pi p_2^*}{r}$, but the last equation implies $p_1 \equiv \pm p_2 \pmod r$, which contradicts our choice of R.

2) Both p_1 and p_2 are not divisible by 4. Take $r = R$. Then $d_i = 2$, and the numbers $c_i = \frac{2p_iR}{4}$ are even, $i = 1, 2$. In this case the two lens space are not distinguishable by Turaev-Viro invariants iff $\cos\frac{\pi {p'}_1^*}{r} = \cos\frac{\pi {p'}_2^*}{r}$, which implies that $p'_1 \equiv \pm p'_2 \pmod R$, and therefore $p_1 \equiv \pm p_2 \pmod R$, contradicting our choice of R.

3) Both p_1 and p_2 and not divisible by 4. Take $r = 2R$. Then $d_1 = d_2 = 2$, and the numbers c_1 and c_2 are odd. Similarly to the above cases, the equality of the invariants implies $p'_1 \equiv \pm p'_2 \pmod{2R}$, and therefore $p_1 \equiv \pm p_2 \pmod R$. Which again contradicts our choice.

In all other cases we can take $r = 3$ and apply Corollary 1.18. □

Proposition 1.20. *If q equals neither 1 nor $p - 1$, then two lens spaces $L_{p,1}$ and $L_{p,q}$ are distinguished by Turaev-Viro invariants.*

Proof. It suffices to set $r = p$. Then $d = (p, 2r) = p$, and $c = \frac{2p^2}{p^2} = 2$ is even. Obviously, $d = p > 2$, therefore, by (1.3), $TV_r(L_{p,1}) = \frac{d}{2r} = \frac{1}{2}$, but $TV_r(L_{p,q}) = 0$. □

Lemma 1.21. *If the lens spaces L_{p,q_1} and L_{p,q_2} are distinguished by Turaev-Viro invariants and p is an even number, then there exists an $r' \geq 3$ such that $TV_{r'}(L_{p,q_1}) \neq TV_{r'}(L_{p,q_2})$ and the number $c' = \frac{2pr'}{gcd^2(p,2r')}$ is even.*

Proof. Let us suppose that the lens spaces L_{p,q_1} and L_{p,q_2} are distinguished by Turaev-Viro invariants for a certain $r \geq 3$. Set $d = \gcd(p, 2r)$ and $c = 2pr/d^2$. Assume that c is odd. Corollary 1.15 implies that $d/2$, and hence r, are even. It follows from (1.3) that $q_1 \equiv \frac{d}{2} \pm 1 \pmod d$, $q_2 \not\equiv \frac{d}{2} \pm 1 \pmod d$. In particular, since q_1 and q_2 are odd, we have $d \neq 4$, i.e., $\frac{d}{2} > 2$.

There are two cases.

1) $q_2 \not\equiv \pm 1 \pmod{\frac{d}{2}}$. Put $r' = \frac{r}{2}(p+1)$. Since c is odd, then $2r/d$ is also odd, and we have $d' = \gcd(p, 2r') = \gcd(p, r) = d/2$, c' is even, $q_1 \equiv \pm 1 \pmod{d'}$, and $q_2 \not\equiv \pm 1 \pmod{d'}$.

2) $q_2 \equiv \pm 1 \pmod{\frac{d}{2}}$. Since $q_2 \not\equiv \pm 1 + \frac{d}{2} \pmod d$, we have $q_2 \equiv \pm 1 \pmod d$. Put $r' = 2r$. Since c is odd, then $\frac{p}{d}$ is also odd, therefore $d' = \gcd(p, 4r) = d$, $c' = 2c$ is even, $q_2 \equiv \pm 1 \pmod{d'}$, and $q_1 \not\equiv \pm 1 \pmod{d'}$.

In both cases the formula (1.3) implies that the invariant $TV_{r'}$ for the two lens spaces is different. □

Proposition 1.22. *Suppose that q_1 and q_2 are not equal to 1. Then two lens spaces L_{p,q_1} and L_{p,q_2} are distinguished by Turaev-Viro invariants iff there exist natural numbers k and v such that*

1) $v > 2$;

2) v is a divisor of p;

3) v is equal to one of the numbers $\frac{q_1\pm 1}{k}$, $\frac{q_2\pm 1}{k}$;

4) If $v=\frac{q_i+1}{k}$, or $v=\frac{q_i-1}{k}$, then $q_j \not\equiv \pm 1 \pmod{v}$, where $\{i,j\}=\{1,2\}$.

PROOF. *Necessity.* Assume that there is an $r \geq 3$ such that $TV_r(L_{p,q_1}) \neq TV_r(L_{p,q_2})$. Then Corollary 1.15 and Lemma 1.21 imply that $d>2$. If p is even then c is also even by Lemma 1.21. If p is odd then c is even by definition. Thus, by (1.3), we may assume that $q_1 \equiv \pm 1 \pmod{d}$, and $q_2 \not\equiv \pm 1 \pmod{d}$. So, there is a natural number k such that $d=\frac{q_1\mp 1}{k}$, and we can put $v=d$.

Sufficiency. If p is odd, then it suffices to put $r=v$. Then $d=\gcd(p,2v)=v>2$, c is even, $q_i \equiv \pm 1 \pmod{d}$, $q_j \not\equiv \pm 1 \pmod{d}$. The formula (1.3) guarantees that the two lens spaces are distinguished by the invariant TV_r. Suppose that p is even. In this case q_1, q_2 are odd, and we can suppose that v is even, because otherwise k is even and one can replace k by $k/2$ and v by $2v$. Indeed, if $q_j \not\equiv \pm 1 \pmod{v}$, then $q_j \not\equiv \pm 1 \pmod{2v}$. Let us consider two cases.

1) The number $\frac{p}{v}$ is odd. Set $r=v$. Then $d=\gcd(p,2v)=v=\frac{q_i\pm 1}{k}$, the number $c=2p/v$ is even, $q_i \equiv \pm 1 \pmod{d}$, and $q_j \not\equiv \pm 1 \pmod{d}$. By (1.3), $TV_r(L_{p,q_1}) \neq TV_r(L_{p,q_2})$.

2) The number $\frac{p}{v}$ is even. If $v=4$, then $4|p$, and $\gcd(p,\frac{p}{2}-1)=1$. In this case, setting $r=p-2$ we get $d=4=v$. If $v>4$, then take $r=\frac{v}{2}$, which again implies that $d=v$. Since $\frac{p}{d}$ is even, c is even too. Thus we reduce this case to the previous one. □

REMARK 1.23. Analyzing the proofs of Propositions 1.19, 1.20, and 1.22, we obtain rough estimates of the value of the parameter r required to distinguish two lens spaces by the invariant TV_r:

1) If $p_1 \neq p_2$, then the lens spaces L_{p_1,q_1} and L_{p_2,q_2} are distinguished by a Turaev-Viro invariant with a certain $r \leq 2R$, where R is the smallest natural number that is coprime with p_1 and p_2 and such that $p_1 \not\equiv \pm p_2 \pmod{R}$.

2) If two lens spaces with the same first index p are distinguished by Turaev-Viro invariants, then they are distinguished by some invariant with $r \leq p$.

COROLLARY 1.24. *If p is prime and q_1, q_2 equal neither 1 nor $p-1$, then two lens spaces L_{p,q_1} and L_{p,q_2} are not distinguished by Turaev-Viro invariants.*

COROLLARY 1.25. *The simplest (in the sense of the value of the first parameter) nonhomeomorphic lens spaces that are not distinguished by Turaev-Viro invariants are $L_{11,2}$ and $L_{11,4}$. The simplest lens spaces that are not distinguished by Turaev-Viro invariants and are not homotopy equivalent are $L_{13,3}$ and $L_{13,5}$.*

REMARK 1.26. If p is prime and lens spaces L_{p,q_1} and L_{p,q_2} are not homeomorphic, then they are distinguished by a certain Reshetikhin-Turaev invariant (see [Je], Remark 3.9).

In conclusion, we will re-formulate Propositions 1.19, 1.20, and 1.22 into one simple criterion. This re-formulation was suggested by S. V. Matveev. In order to do that we need to introduce a function $h_v\colon \mathbf{Z} \to \mathbf{Z}_2$, for every positive integer v, such that

$$h_v(x) = \begin{cases} 1, & \text{if } x \equiv \pm 1 \pmod v; \\ 0, & \text{if } x \not\equiv \pm 1 \pmod v. \end{cases}$$

THEOREM 1.27.(Criterion of TV-equivalence) *Two lens spaces L_{p_1,q_1} and L_{p_2,q_2} are not distinguished by Turaev-Viro invariants iff $p_1 = p_2 = p$, and for any $v > 2$ such that p is divisible by v we have $h_v(q_1) = h_v(q_2)$.*

PROOF. *Necessity.* By Proposition 1.19, $p_1 = p_2$. Assume that there is $v > 2$ such that $v|p$, $h_v(q_1) = 1$, and $h_v(q_2) = 0$. In this case there is an integer $k \geq 0$ such that $q_1 = \pm 1 + kv$. If $k = 0$ then we get a contradiction with Proposition 1.20, if $k > 0$ then we get a contradiction with Proposition 1.22.

Sufficiency. Assume that $p_1 = p_2$ and for any divisor $v > 2$ of p we have $h_v(q_1) = h_v(q_2)$. Then q_1 and q_2 can not be equal neither 1 nor $p - 1$, because otherwise $h_v(q_1) \neq h_v(q_2)$. Therefore, we can apply Proposition 1.22. Our conditions contradict existence of positive numbers v and k such that $v > 2$, $v|p$, $v = \frac{q_i \pm 1}{k}$, and $q_i \not\equiv \pm 1 \pmod v$, where $i, j = 1, 2, \quad i \neq j$. $\square$

CHAPTER 2
SURGERIES ON PERIODIC LINKS AND HOMOLOGY OF PERIODIC 3-MANIFOLDS

In the early 1960's both Wallace [Wa] and Lickorish [Li1] proved that every closed, connected, orientable 3-manifold may be obtained by surgery on a framed link in S^3. Thus, link diagrams may be used to depict manifolds. Every manifold has infinitely many different framed link descriptions. However, in 1970's Kirby [K] showed that two framed links determine the same 3-manifold iff they are related by a finite sequence of two specific types of moves. This calculus of framed links, together with the earlier results, gives a classification of 3-manifolds in terms of equivalence classes of framed links. The framed link representation of 3-manifolds has proven to be extremely useful. For instance, most of the new 3-manifold invariants originating from famous Witten's paper [Wi] are based on the framed link approach. Therefore, it is always very useful to have some kind of correspondence between certain classes of 3-manifolds and some classes of framed links. One example of such correspondence is the classical relationship between the lens spaces and the chain-link diagrams (see, for instance, [Ro]). This result, in particular, allowed L. Jeffrey to determine exact formulas for the Witten-Reshetikhin-Turaev invariants of the lens spaces [Je]. Another example is the fact that a closed oriented 3-manifold is an integral homology 3-sphere iff it can be obtained by surgery on an algebraically split link with framing numbers ± 1 (see [Mu1], [O1]). This relationship plays a key role in many papers on quantum and finite invariants of integral homology 3-spheres (see, for instance, [O1], [O2], [Mu1], [Mu2]). In Section 1 we will establish an analogous relationship between periodic 3-manifolds and periodic links. Namely, we prove the following theorem:

THEOREM 2.1. *Let p be a prime integer and M be a closed oriented 3-manifold. There is an action of the cyclic group $\mathbf{Z}_p$ on M with the fixed-point set equal to a circle if and only if there exists a framed p-periodic link $L \subset S^3$ such that M is the result of surgery on L and $\mathbf{Z}_p$ acts freely on the set of components of L.*

A special case of Theorem 2.1 when M is a homology sphere was proven in [KB-Pr]. In the general form the theorem was proven for the first time by Józef Przytycki in his graduate course *Topics in Algebra Situs*[1]. A similar result is obtained for manifolds with free $\mathbf{Z}_p$ actions:

THEOREM 2.2. *Let p be a prime integer and M be a closed oriented 3-manifold. There is a free action of the cyclic group $\mathbf{Z}_p$ on M iff there exists a framed p-periodic*

[1]The George Washington University, February of 1999.

link $L \subset S^3$ admitting a free action of $\mathbf{Z}_p$ on the set of its components such that M is the result of surgery on $L' = L \cup \gamma$, where γ is the axis of the action with framing co-prime to p.

In Section 2 we give an interesting application of Theorem 2.1. Namely, we prove the following result:

THEOREM 2.3. *If a closed orientable 3-manifold M admits an action of a cyclic group $\mathbf{Z}_p$ where p is an odd prime integer and the fixed point set of the action is S^1 then $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$.*

Note that this theorem provides a non-trivial criterion for 3-manifolds admitting the described action. The simplest examples of 3-manifolds with $H_1(M; \mathbf{Z}_p) = \mathbf{Z}_p$ are lens spaces $L_{pn,q}$ (or more generally, (pn/q) Dehn surgeries on knots in S^3).

Theorem 2.3 was first announced as a conjecture[2] and partially proven[3] in April of 1999 [S9] (It is interesting to mention that the conjecture was influenced by the study of Murakami-Ohtsuki-Okada invariants on periodic 3-manifolds[4], but the equation $MOO_p(M) = \pm G_p^{rkH_1(M;\mathbf{Z}_p)}$ eventually led to the more "classical" algebraic topology. Here p is an odd prime integer, MOO_p is the Murakami-Ohtsuki-Okada invariant parameterized by $q = e^{2\pi i/p}$, and $G_p = \sum_{j \in \mathbf{Z}_p} q^{j^2}$, see Theorem 3.4). Recently (November, 1999), Adam Sikora announced a proof of the theorem [Si]. In fact, using some classical but involved algebraic topology, he obtained more general results implying our theorem. The surgery presentation of periodic 3-manifolds developed in Section 1 allowed us to find an elementary proof of Theorem 2.3, presented in Section 2.

Theorem 2.3 is not true for $p = 2$, see Remark 2.20. An interesting criterion for 2-periodic rational homology spheres is provided by the following theorem.

THEOREM 2.4. *Let M be a rational homology 3-sphere such that the group $H_1(M; \mathbf{Z})$ does not have elements of order 16. If M admits an orientation preserving action of $\mathbf{Z}_2$ with the fixed point set being a circle then the canonical decomposition of the group $H_1(M; \mathbf{Z})$ has even number of terms $\mathbf{Z}_2$ and even number of terms $\mathbf{Z}_4$, and arbitrary number of terms $\mathbf{Z}_8$.*

This chapter in the main part coincides with our joint paper with Józef Przytycki [PrS].

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[2]The conjecture was obtained as a result of extensive computations performed with a program written in *Mathematica*.

[3]In the case when the orbit space of the action can be obtained from S^3 by an integer surgery on a knot.

[4]In turn, our interest in Murakami-Ohtsuki-Okada invariants was sparked by their relation with the q-analog of the first homology skein module, see Chapter 4.

2.1 Periodic 3-manifolds are surgeries on periodic links

We show in this section that p-periodic closed oriented 3-manifolds can be presented as results of integer surgeries on p-periodic links. We show also an analogous result for manifolds with free action of $\mathbf{Z}_p$.

Before we prove theorems 2.1 and 2.2, we need to establish some basic terminology and preliminary lemmas.

2.1.1 Periodic Links

DEFINITION. By a *framed knot* K we mean a ring $S^1 \times [0, \varepsilon]$ embedded in S^3. By the *framing* of K we mean an integer defined as follows. Let V_ε be the ε-neighborhood of $K_0 = S^1 \times \{0\}$, then $K_\varepsilon = S^1 \times \{\varepsilon\}$ is a projection of K_0 onto ∂V_ε. Let P be the projection of K_0 onto ∂V_εwhich is homologically trivial in $S^3 - V_\varepsilon$. The framing f is defined as the algebraic number of intersections of K_ε and P. A *framed link* is a collection of non-intersecting framed knots. We will adopt the usual "blackboard" convention for framed link diagrams.

DEFINITION. A (framed) link L in S^3 is called *p-periodic* if there is a $\mathbf{Z}_p$-action on S^3, with a circle as a fixed point set, which maps L onto itself, and such that L is disjoint from the fixed point set. Furthermore, if L is an oriented link, one assumes that each generator of $\mathbf{Z}_p$ preserves the orientation of L or changes it to the opposite one.

By the positive solution of Smith Conjecture ([MB], [Th]) we know that the fixed point set of the action of $\mathbf{Z}_p$ is an unknotted circle and the action is conjugate to an orthogonal action on S^3. In other words, if we identify S^3 with $\mathbf{R}^3 \cup \infty$, then the fixed point set can be assumed to be equal to the "vertical" axis $z = 0$ together with ∞, and a generator φ of $\mathbf{Z}_p$ can be assumed to be the rotation $\varphi(z,t) = (e^{2\pi i/p} \cdot z, t)$, where the coordinates on $\mathbf{R}^3$ come from the product of the complex plane and the real line $\mathbf{C} \times \mathbf{R}$. Thus, any (framed) p-periodic link L^p may be represented by a φ-invariant diagram, *p-periodic diagram* (with framing parallel to the projection plane), see Fig. 2.1.

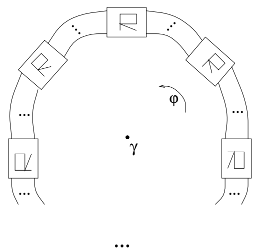


Fig. 2.1

By the *underlying link* for L^p we will mean the orbit space of the action, that is $L_* = L^p/\mathbf{Z}_p$.

LEMMA 2.5. *Let p be a prime integer and $L^p \subset S^3$ be a (framed) p-periodic link. The following three conditions are equivalent:*
1) $\mathbf{Z}_p$ acts freely on the set of components of L^p;
2) The linking number of each component of the underlying link L_* with the axis of rotation is congruent to zero modulo p;
3) The number of components of L^p is p times greater than the number of components of L_*.
PROOF. The equivalence of the conditions 1) and 3) is obvious. To prove that 2) is equivalent to 3), consider the covering projection $L^p \to L_*$. Let l be a component of L_*. The preimage ρ^{-1} of the closed path λ which traverses l exactly once (i.e., l with a base point) consists of p paths $\lambda_1, \ldots, \lambda_p$ in L^p. The condition $\operatorname{lk}(l,\gamma) = 0 \pmod p$ is equivalent to the condition that each of λ_i is closed. Thus, l lifts to p components in L^p iff $\operatorname{lk}(l,\gamma) = 0 \pmod p$. □

DEFINITION. A p-periodic link L^p that satisfies any of the conditions from Lemma 2.5 will be called *strongly p-periodic.*

2.1.2 Periodic Manifolds

DEFINITION. A 3-manifold M is called *p-periodic* if it admits an orientation preserving action of the cyclic group $\mathbf{Z}_p$ with a circle as a fixed point set, and the action is free outside the circle.

We can show immediately the easy part of Theorem 2.1. Indeed, consider a strongly p-periodic framed link L^p, and let M be the 3-manifold obtained by surgery on L^p. By definition of a framed p-periodic link, there is a $\mathbf{Z}_p$ action on S^3, and on $S^3 - L^p$, with a circle γ as a fixed point set. This action induces a $\mathbf{Z}_p$ action on M. Moreover, since the action of $\mathbf{Z}_p$ is free on the set of components of L^p, there are no other fixed points of the action of $\mathbf{Z}_p$ on M but the circle γ.

To show the difficult part of Theorem 2.1 we first fix some notation. Suppose that $\mathbf{Z}_p$ acts on M with the fixed-point set equal to a circle γ. Denote the quotient by $M_* = M/\mathbf{Z}_p$, the projection map by $h: M \to M_*$ and $\gamma_* = h(\gamma)$.

LEMMA 2.6. *The map $h_*: H_1(M) \to H_1(M_*)$ is an epimorphism.*
PROOF. Let $x_0 \in \gamma$. Since x_0 is a fixed point of the action, any loop based at $h(x_0)$ lifts to a loop based at x_0. Thus $h_\# : \pi_1(M, x_0) \to \pi_1(M_*, h(x_0))$ is an epimorphism, and since H_1 is an abelianization of π_1, the map h_* is also an epimorphism. □

Notice that the proof works for any finite group action on a manifold with a non-empty fixed point set.

Let us recall the Lefschetz' duality theorem which we will use in our proof of Theorem 2.1. First some terminology: a compact connected n-dimensional manifold

M is called *R-oriented* for a commutative ring with identity R, if $H_n(M, \partial M; R) = R$. In particular, any manifold is $\mathbf{Z}_2$-oriented, and an oriented manifold is R-oriented for any ring R. For a reference, see [Sp].

THEOREM 2.7. (Lefschetz) *Let M be a compact n-dimensional, R-oriented manifold. Then there is an isomorphism $\tau : H^q(M; R) \to H_{n-q}(M, \partial M; R)$. Furthermore if R is a PID (principal ideal domain) and $H_{q-1}(M, R)$ is free then $H^q(M; R) = Hom(H_q(M; R), R)$ and for $\alpha \in H^q(M; R)$ and $c \in H_q(M; R)$ one has: $\alpha(c) = alg(c, \tau(\alpha))$, where $alg(c, \tau(c)) \in R$ is the algebraic intersection number of c and $\tau(\alpha)$ in M (alg: $H_q(M; R) \times H_{n-q}(M, \partial M; R) \to R$).*

We use Lefschetz' Theorem to show that the covering $h : M \to M_*$ is yielded by a 2-chain whose boundary is a multiple of γ_*. Because we work with $q = 1$, then $H_{q-1}(M, R)$ is free and we can use the intersection number interpretation of the Lefschetz' Theorem.

LEMMA 2.8. *Let M be a closed orientable p-periodic 3-manifold. With the notations as before, one has:*

(1) $\gamma_* \equiv 0$ in $H_1(M_*, \mathbf{Z}_p)$.

(2) There is a 2-chain $C \in C_2(M_*, \mathbf{Z}_p)$ such that $\partial C \equiv m\gamma_*$ mod p and the covering $h : (M - \gamma) \to (M_* - \gamma_*)$ is yielded by the map $\phi_C : H_1(M_* - \gamma_*) \to \mathbf{Z}_p$ where $\phi_C(K)$ is the intersection number of K with C (i.e. for a 1-cycle $K \in M_* - \gamma_*$, $\phi_C(K) = alg(K, C)$ where $alg(K, C)$ is the intersection number of K with C, well defined mod p)[5]. In particular $\phi_C(\mu_*) = m$, where μ_* is a meridian of γ_*.

PROOF. To work with Lefschetz' Theorem we have to consider compact manifolds. Thus, instead of $M_* - \gamma_*$ we consider a homotopically equivalent compact manifold $\hat{M}_* = M_* - int(V_{\gamma_*})$, where V_{γ_*} is a regular neighborhood of γ_* in M_*. Similarly, let $V_\gamma = h^{-1}(V_{\gamma_*})$ be a Z_p-invariant regular neighborhood of γ in M. Let also $\hat{M} = M - int(V_\gamma)$. Since $\hat{h}\colon \hat{M} \to \hat{M}_*$ is a regular covering, it is characterized by an epimorphism $\pi_1(\hat{M}_*) \to \pi_1(\hat{M}_*)/\pi_1(\hat{M}) = \mathbf{Z}_p$ (up to an automorphism of $\mathbf{Z}_p$). Thus, since $\mathbf{Z}_p$ is abelian, $\hat{h}$ is defined by an epimorphism $\phi : H_1(\hat{M}_*) \to \mathbf{Z}_p$, where ϕ is unique up to an automorphism of $\mathbf{Z}_p$. Let $\hat{C}$ be a 2-cycle representing the element of $H_2(\hat{M}_*, \partial\hat{M}_*; \mathbf{Z}_p)$ dual to the epimorphism ϕ, that is, such that $alg(K, \hat{C}) \equiv \phi(K)$ (mod p) for any $K \in H_1(\hat{M}_*)$. We can assume that $\hat{C} \cap \partial\hat{M}_*$ is a collection of simple noncontractible curves in the torus $\partial\hat{M}_*$. Finally let C be a 2-chain obtained from $\hat{C}$ by adding to $\hat{C}$ annuli connecting the components of $\hat{C} \cap \partial\hat{M}_*$ with γ_* in V_{γ_*}. Thus, C of part (2) is constructed.

Let μ_* be a meridian of γ_* (or more precisely, of V_{γ_*}). $\phi(\mu_*) \neq 0 \pmod p$ because γ_* is a branching set of the covering, so the preimage of μ_* (under h) is a connected

[5]If $H_2(M_*, \mathbf{Z}) = 0$, then $alg(K, C) = lk(K, m\gamma_*)$, but generally $alg(K, C)$ depends on the choice of C.

curve μ, (a meridian of γ in M), by the definition of a branched covering. Thus, there is $0 < m < p$ such that $\phi(\mu_*) = m$. We can conclude also that $m\gamma_* \equiv \partial C \bmod p$, thus $\gamma_* \equiv 0$ in $H_1(M_*; \mathbf{Z}_p)$. □

Proof of Theorem 2.1. By the classical result of Wallace (1960) and Lickorish (1962), every closed oriented 3-manifold is a result of a surgery on a framed link in S^3. In particular, M_* can be represented as a result of surgery on some framed link $L_\#$ in S^3. Inversely S^3 can be obtained as a result of surgery on some framed link $\hat{L}_\#$ in M_*. We can assume that $\hat{L}_\#$ satisfies the following conditions (possibly after deforming $\hat{L}_\#$ by ambient isotopy):

(1) $\gamma_* \cap \hat{L}_\# = \emptyset$;

(2) $alg(\hat{L}^i_\#, C) \equiv 0 \ mod\ p$, for any component $\hat{L}^i_\#$ of $\hat{L}_\#$.

$\hat{L}_\#$ satisfying the conditions 1-2 can be obtained as follows:
Let $L_\# \subset S^3$ be a framed link in S^3 such that M_* is a result of surgery on $L_\#$. Let $\hat{L}_\#$ denote the co-core of the surgery [6]. In particular, $\hat{L}_\#$ is a framed link in M_* such that S^3 is a result of surgery on $\hat{L}_\#$. By a general position argument, we can make γ_* and $\hat{L}_\#$ disjoint, but in order to get condition (2) we should do so in a controllable manner.

Let C be the 2-chain from Lemma 2.8. Let $\hat{L}^i_\#$ be any component of $\hat{L}_\#$. If we change a crossing between $\hat{L}^i_\#$ and γ_* then the algebraic crossing number, $alg(\hat{L}^i_\#, C)$ changes by $\pm m$ mod p. Thus, by a series of crossing changes we can get $alg(\hat{L}^i_\#, C) \equiv 0 \ mod\ p$ for any component of $\hat{L}_\#$, providing condition (2).

This implies that we can easily modify C (outside γ_*) so that $C \cap \hat{L}_\# = \emptyset$. Therefore, C survives the surgery (as well as γ_*), and in S^3 it has $\mathbf{Z}_p$ boundary $m\gamma_*$ and it is disjoint from $L_\#$ (link in S^3 being the co-core of the surgery on $\hat{L}_\#$ in M_*). Thus, $\mathrm{lk}(L^i_\#, \gamma_*) \equiv 0 \ mod\ p$ for any component $L^i_\#$ of $L_\#$.

Now we are ready to unknot γ_* using Kirby calculus ([K], [FR]). Choose some orientation on γ_*. We can add unlinked components with framing ± 1 to $L_\#$ around each crossing of γ_*, making sure that arrows on γ_* run opposite ways (i.e. the linking number of γ_* with the new component is zero, see Fig. 2.2). Use the K-move to change the appropriate crossings and thus to unknot γ_*. Thus we trivialized γ_* without compromising conditions (1) and (2). Denote the framed link obtained from the initial link $L_\#$ after the described isotopy and adding the new components by L_*.

[6] In our notation, *core* of the surgery is the framed surgery link, that is the framed link, regular neighborhood of which is removed in the "drilling" part of the surgery, with framing yielded by the meridian of the attached ("filling" part of the surgery) solid torus. The *co-core* of the surgery is the core of the "filling" solid torus, with its framing yielded by the meridian of the removed solid torus. The surgery on the co-core link brings back the initial manifold.

Notice, that each new component that we introduce during the above procedure has linking number 0 with any other component of L_*.

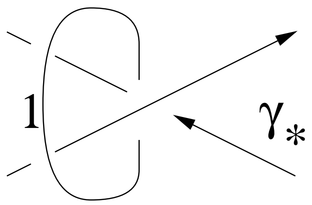


Fig. 2.2

To complete the proof of the theorem, consider the p-fold cyclic branched covering of S^3 by S^3 with branching set γ_*. Let L denote the preimage of L_*. Notice that L is strongly p-periodic. We claim that the result of performing surgery on L is $\mathbf{Z}_p$-homeomorphic to M.

The preimage of each component of L_* consists of p components permuted by a $\mathbf{Z}_p$ action, by Lemma 2.5. Therefore, $\mathbf{Z}_p$ acts on the result of the surgery on S^3 along L, $\widetilde{M} = (S^3, L)$, with a branched set γ and quotient $M_* = (S^3, L_*)$.

The group $H_1(M_* - \gamma_* - \hat{L}_\#) = H_1(S^3 - \gamma_* - L_*)$ is generated by μ_* (a meridian of γ_*) and meridians of the components of L_*, say $b_1, \ldots, b_k$. Of course, $H_1(M_* - \gamma_*)$ is also generated by $\mu_*, b_1, \ldots, b_k$. By Lemma 2.8, the covering $\rho\colon (M - \gamma) \to (M_* - \gamma_*)$ is characterized by the map $\phi_C\colon H_1(M_* - \gamma_*) \to \mathbf{Z}_p$ (up to an automorphism of $\mathbf{Z}_p$), where $\phi_C(K)$ is the intersection number of $K \in H_1(M_* - \gamma_*)$ with C modulo p. Similarly, the covering $\widetilde{\rho}\colon (\widetilde{M} - \gamma) \to (M_* - \gamma_*)$ is characterized by a map $\phi_2\colon H_1(M_* - \gamma_*) \to \mathbf{Z}_p$. By our construction, $\phi_C(\mu_*) = m$ and $\phi_C(b_i) = 0$ for every $1 \leq i \leq k$. We need to show that $\phi_2(\mu_*) = m'$ for some m' coprime to p, and $\phi_2(b_i) = 0$ for every $1 \leq i \leq k$. This follows from the fact that $\widetilde{\rho}^{-1}(b_i)$ consists of p loops and $\widetilde{\rho}^{-1}(\mu_*)$ is a single loop. Thus, ϕ_2 and ϕ_C are equivalent up to the automorphism of $\mathbf{Z}_p$ sending m' to m. Therefore, the manifolds $(\widetilde{M} - \gamma)$ and $(M - \gamma)$ are $\mathbf{Z}_p$-homeomorphic, with a homeomorphism given by $g\colon (\widetilde{M} - \gamma) \to (M - \gamma)$ such that $\widetilde{\rho} = \rho \circ g$. Notice, that $\widetilde{M}$ can be obtained from $(\widetilde{M} - \gamma)$ by attaching a 2-handle along $\widetilde{\rho}^{-1}(\mu_*)$ and then a 3-handle, and M can be obtained from $(M - \gamma)$ by attaching a 2-handle along $\rho^{-1}(\mu_*)$ and then a 3-handle. Since $g(\widetilde{\rho}^{-1}(\mu_*)) = \rho^{-1}(\mu_*)$, the homeomorphism g can be extended to a $\mathbf{Z}_p$-homeomorphism $\hat{g}\colon \widetilde{M} \to M$. Our proof of the Theorem 2.1 is completed. □

Remark 2.9. Notice, that if in the proof of Theorem 2.1 we assumed that the link $L_\#$ was algebraically split then the link L_* would be algebraically split as well. This remark will be important later in the proofs of Theorems 2.3 and 2.4.

As a corollary we obtain a proof of Theorem 2.2.

Proof of Theorem 2.2. Consider any $\mathbf{Z}_p$ equivariant knot, say $\hat{\gamma}$ in M. Let $V_{\hat{\gamma}}$ be a $\mathbf{Z}_p$ equivariant regular neighborhood of $\hat{\gamma}$ in M and γ' a curve on $\partial V_{\hat{\gamma}}$ which is also $\mathbf{Z}_p$ equivariant. Notice, that γ' intersects a meridian of $V_{\hat{\gamma}}$ exactly once. Now let M' be a manifold obtained from M by a surgery on $\hat{\gamma}$ with the framing defined by γ'. Let $\gamma \in M'$ be the co-core of the surgery. The $\mathbf{Z}_p$ action on M yields the action on M' and our choice of framing guarantees that γ is the (only) fixed point set of the action. Thus, we can apply to M' the previous theorem. This proves that M can be obtained by an integer surgery on $L \cup \gamma$, where L is a strongly p-periodic link. Furthermore, the framing of γ must be coprime to p, to insure that $\mathbf{Z}_p$ acts on the resulting manifold with no fixed points. □

REMARK 2.10. We plan to extend Theorem 2.1 to any $\mathbf{Z}_p$ orientation preserving action on a closed 3-manifold M, and to $\mathbf{Z}_{p^k}$ actions.

2.2 Homology of periodic 3-manifolds

The main goal of this section is to give elementary proofs of Theorems 2.3 and 2.4 using the surgery presentation of p-periodic 3-manifolds developed in Section 1.

2.2.1 Linking matrices of framed strongly p-periodic links and of algebraically split links

Let L be a framed oriented link of n components $l_1, \ldots, l_n$. The *linking matrix* of L is the matrix $(a_{ij})_{n\times n}$ defined by

$$a_{ij} = \begin{cases} \mathrm{lk}(l_i, l_j) & \text{if } i \neq j \\ \text{framing of } l_i & \text{if } i = j \end{cases}$$

Let L^p be a framed strongly p-periodic link and L_* be the corresponding underlying link. Fix an orientation of L_* and denote the components of L_* by $l_1, \ldots, l_n$. Consider a p-periodic diagram of L^p. Denote the p copies of the tangle R from the diagram (see Fig. 2.1) by $R_1, \ldots, R_p$ in the clockwise order. Lift the orientation of L_* to L^p. By Lemma 2.5, each component l_i of L_* has p covering preimages in L^p. Denote them in a clockwise order by $l_{i1}, \ldots, l_{ip}$. By the clockwise order here we mean such an order that if we choose any point $x \in l_{ij} \cap R_j$ then the corresponding point in R_{j+1} will belong to l_{ij+1} (subscripts are treated modulo p).

Now consider the following natural order for the components of L^p:

$$l_{11}, \ldots, l_{1p}, l_{21}, \ldots, l_{2p}, \ldots, l_{n1}, \ldots, l_{np}.$$

It is not hard to see that with regard to this order, the linking matrix A_p for L^p is of the following form

$$A_p = \begin{pmatrix} A_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & A_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & A_{nn} \end{pmatrix},$$

where all the blocks are $p \times p$, A_{ii} is the linking matrix for the sublink consisting of $l_{i1}, \dots, l_{ip}$, and B_{ij} is the matrix with elements $b^{ij}_{ks} = \mathrm{lk}(l_{ik}, l_{js})$.

Recall, that a matrix $(a_{ij})_{k\times k}$ is called *circulant* if $a_{ij} = a_{i+1j+1}$, $i, j = 1, \dots, k$ (subscripts mod k).

PROPOSITION 2.11. *Every block in A_p is a circulant matrix.*
PROOF. Consider $B_{ij} = (b^{ij}_{ks})_{p\times p}$. Then $b^{ij}_{ks} = \mathrm{lk}(l_{ik}, l_{js})$ and $b^{ij}_{k+1s+1} = \mathrm{lk}(l_{ik+1}, l_{js+1})$. If one rotates the p-periodic diagram of L^p around the center in the clockwise direction by $2\pi/p$ then the pair (l_{ik}, l_{js}) will go into (l_{ik+1}, l_{js+1}), taking subscripts modulo p. Thus, $\mathrm{lk}(l_{ik}, l_{js}) = \mathrm{lk}(l_{ik+1}, l_{js+1})$. If we notice that the framing numbers of $l_{i1}, \dots, l_{ip}$ are all the same, then the above argument shows that A_{ii} is also circulant for any $i = 1, \dots, n$. □

DEFINITION. We will call a (framed) link L *algebraically split* if the linking number between any two components of L is zero. A strongly p-periodic (framed) link L^p will be called *orbitally separated* if the underlying link L_* is algebraically split.

REMARK 2.12. It is not hard to see that a strongly p-periodic link L^p is orbitally separated iff any two components of L^p that cover different components of L_* have the linking number equal to zero.

COROLLARY 2.13. *It follows from Proposition 2.11 and Remark 2.12 that L^p is an orbitally separated link iff all the non-diagonal blocks B_{ij} in A_p are zero matrices.* □

2.2.2 Nullity of symmetric circulant matrices over $\mathbf{Z}_p$

Circulant matrices are very well studied and a lot is known about them (see, for instance, [Da]). But, apparently, not much is known about circulant matrices over finite fields (or rings). The following two results provide the key tool for our proof of Theorem 2.3, but they also appear to be interesting from a purely matrix theoretical point of view.

LEMMA 2.14. *If*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$$

is a circulant matrix with integer elements then

$$det\ A = \begin{cases} a_1^n + a_2^n + \ldots + a_n^n \pmod{n}, & \textit{if } n \textit{ is odd;} \\ a_1^n - a_2^n + \ldots - a_n^n \pmod{n}, & \textit{if } n \textit{ is even.} \end{cases}$$

PROOF. The determinant of A is a sum of $n!$ terms. The terms of the form a_i^n, $i = 1, \ldots, n$, will be called *diagonal.* Note that any term different from diagonal appears in the sum exactly n times:

$$a_{i_1} a_{i_2} a_{i_3} \ldots a_{i_n},$$

$$a_{i_2} a_{i_3} \ldots a_{i_n} a_{i_1},$$

$$\vdots$$

$$a_{i_n} a_{i_1} a_{i_2} \ldots a_{i_{n-1}}.$$

Note that the sign for all such terms is the same. To prove this we need to show that the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix} \text{ and } \sigma' = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_n + 1 & i_1 + 1 & i_2 + 1 & \cdots & i_{n-1} + 1 \end{pmatrix}$$

have the same parity (everything is modulo n). Obviously,

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_n + 1 & i_1 + 1 & i_2 + 1 & \cdots & i_{n-1} + 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & \cdots & n & 1 \\ i_1 + 1 & i_2 + 1 & \cdots & i_{n-1} + 1 & i_n + 1 \end{pmatrix}.$$

The row $(\,2 \quad 3 \quad 4 \quad \cdots \quad n \quad 1\,)$ has $n-1$ inversions. The numbers of inversions in $(\,i_1 \quad i_2 \quad \cdots \quad i_n\,)$ and $(\,i_1 + 1 \quad i_2 + 1 \quad \cdots \quad i_n + 1\,)$ also differ by $n-1$. Thus the parities of σ and σ' are the same. Therefore, the total sum of all non-diagonal terms in det A is 0 mod n. The result follows. □

Denote the nullity of A over $\mathbf{Z}_n$ by $null_n A$.

LEMMA 2.15. *Let p be an odd prime integer and A be a $p \times p$ symmetric circulant matrix over $\mathbf{Z}_p$, then $null_p A \neq 1$.*

PROOF. Assume $null_p A > 0$, i. e. $a_1^p + 2a_2^p + \ldots + 2a_{\frac{p+1}{2}}^p = a_1 + 2a_2 + \ldots + 2a_{\frac{p+1}{2}} = 0$ (mod p), by Lemma 2.14 (the second equality follows from Fermat's theorem). After adding all rows to the last one and all columns to the last column we get

$$det\ A = det \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_3 & 0 \\ a_2 & a_1 & a_2 & \cdots & a_4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_3 & a_4 & a_5 & \cdots & a_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Denote the ith column of the above matrix by C_i. Then the linear combination

$$(p-1)C_1 + (p-2)C_2 + \ldots + 2C_{p-2} + C_{p-1}$$

is 0 modulo p. Indeed, the ith row of the linear combination is

$$(p-1)a_i + (p-2)a_{i+1} + \ldots + 2a_{i-3} + a_{i-2},$$

all the coefficients and subscripts are modulo p. It is not hard to see that after the substitution $a_1 = -2a_2 - 2a_3 - \ldots - 2a_{\frac{p+1}{2}}$, the coefficient for a_k $(k \neq 1)$ in the above sum is

$$-2(p-i) + (p-(i+k)) + (p-(i-k)) = 0.$$

Thus, if $det\ A = 0$ over $\mathbf{Z}_p$ then $null_p A \geq 2$. □

REMARK 2.16. Lemma 2.15 is not true if $p = 2$. For instance, nullity of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is 1 over $\mathbf{Z}_2$.

2.2.3 Proof of Theorem 2.3

In this section we will prove Theorem 2.3, the main theorem of Section 2. Let M be a closed oriented 3-manifold obtained by a Dehn surgery on a framed oriented link L, and let A be the linking matrix of L. The following fact is well-known.

LEMMA 2.17. *$null_p A = rank\ H_1(M; \mathbf{Z}_p)$.* □

Now we are ready to prove an important special case of Theorem 2.3.

PROPOSITION 2.18. *Let p be an odd prime integer. If a closed orientable 3-manifold M can be obtained from S^3 by Dehn surgery on an orbitally separated framed link L^p then $H_1(M; \mathbf{Z}_p) \neq \mathbf{Z}_p$.*

PROOF. Assume that M can be obtained by Dehn surgery on an orbitally separated framed link L^p. Let A_p be the linking matrix of L^p, as constructed in Section 2.1. By Corollary 2.13, A_p is a block diagonal matrix. Therefore, $null_p A_p$ is equal to the sum $null_p A_{11} + \ldots + null_p A_{nn}$. By Proposition 2.11, each A_{ii} is a circulant matrix. Moreover, since A_p is a linking matrix, each A_{ii} is symmetric. By Lemma 2.15,

$null_pA_{ii} \neq 1$, hence, $null_pA_p \neq 1$. It follows from Lemma 2.17, that $H_1(M;\mathbf{Z}_p) \neq \mathbf{Z}_p$. □

To finish our proof of the main theorem of Section 2 we need the following result (Corollary 2.3 in [Mu2]).

PROPOSITION 2.19. (H. Murakami) *Fix an odd prime r. For every connected, closed, oriented 3-manifold M, there exist lens spaces $L(n_1,1),\ldots,L(n_k,1)$ with n_i coprime to r such that the connected sum $M\#L(n_1,1)\#\ldots\#L(n_k,1)$ can be obtained by Dehn surgery on an algebraically split link with integer framing.* □

Now we are ready to prove Theorem 2.3 in full generality.

PROOF OF THEOREM 2.3. Let M be a p-periodic closed oriented 3-manifold and $M_* = M/\mathbf{Z}_p$. By Proposition 2.19, there are integers $n_1,\ldots,n_k$ coprime to p such that the connected sum $\widetilde{M_*} = M_*\#L(n_1,1)\#\ldots\#L(n_k,1)$ can be obtained by Dehn surgery on an algebraically split framed link $\widetilde{L_*}$. Consider $\widetilde{M} = M\#pL(n_1,1)\#\ldots\#$ $\#pL(n_k,1)$. Obviously, $\widetilde{M}$ is p-periodic such that $\widetilde{M_*} = \widetilde{M}/\mathbf{Z}_p$. Moreover, it easily follows from the proof of Theorem 2.1 that $\widetilde{M}$ can be obtained using Dehn surgery on an orbitally separated framed link (see Remark 2.9). Therefore, by Proposition 2.18, $H_1(\widetilde{M};\mathbf{Z}_p) \neq \mathbf{Z}_p$. Since the numbers $n_1,\ldots,n_k$ are coprime to p, we have $H_1(L(n_i,1),\mathbf{Z}_p) = 0$ for every i. This implies that $H_1(M;\mathbf{Z}_p) = H_1(\widetilde{M};\mathbf{Z}_p) \neq \mathbf{Z}_p$. □

2.2.4 Orientation preserving $\mathbf{Z}_2$ actions. Proof of Theorem 2.4.

REMARK 2.20. Theorem 2.3 is not true in the case $p = 2$. A simple counterexample is $S^2 \times S^1$. It is interesting to notice that $S^2 \times S^1$ admits two different orientation preserving actions of $\mathbf{Z}_2$ with the fixed point set being a circle. Indeed, let $H(1,1)$ be the negative Hopf link with framing 1 on each component and let $H(-1,-1)$ be the negative Hopf link with framing -1 on each component (see Fig. 2.3). Dehn surgery on each of these framed links produces $S^2 \times S^1$. Moreover, permutation of the components defines $\mathbf{Z}_2$ actions on $S^2 \times S^1$ with a circle as the fixed point set. These two actions are different. In the first case the orbit space of the action M_* is $S^2 \times S^1$, and in the second case $M_* = RP^3$. Furthermore, one can see that in the first case $\mathbf{Z}_2$ acts on $\mathbf{Z} = H_1(S^2 \times S^1)$ trivially, and in the second case it sends 1 to -1. Both of the described actions were studied in [Pr3].

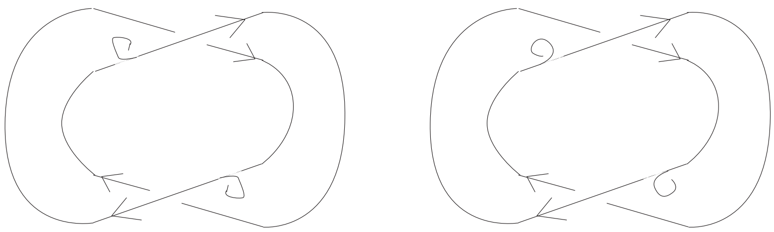


Fig. 2.3

An interesting criterion for 2-periodic rational homology 3-spheres is provided by Theorem 2.4. Before we prove it, let us recall that every finite abelian group can be uniquely decomposed into a direct sum of cyclic groups whose orders are powers of prime numbers. Such decomposition will be called *canonical*.

PROOF OF THEOREM 2.4. Let M be a rational homology 3-sphere such that $H_1(M, \mathbf{Z})$ does not have elements of order 16. Assume that M admits an orientation-preserving action of $\mathbf{Z}_2$ such that the fixed point set is a circle. As before, let $M_* = M/\mathbf{Z}_2$ be the orbit space of the action. By Theorem 2.1, M can be obtained using Dehn surgery on a strongly 2-periodic framed link L^2 with the underlying link L_*. By Lemma 2.6, M_* is also a rational homology 3-sphere. Therefore, M_* can be obtained by surgery on an algebraically split framed link (see [Mu1], [O1]). Thus, we may assume that L_* is algebraically split (see Remark 2.9). By definition, L^2 is orbitally separated, and by Proposition 2.11 and Corollary 2.13 the linking matrix of L^2 is block diagonal with every block being a 2×2 symmetric circulant matrix. Recall that the linking matrix of L^2 can be considered as a presentation matrix for the abelian group $H_1(M, \mathbf{Z})$. Therefore, it is enough to show that every finite abelian group G presented by a matrix $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a, b \in \mathbf{Z}$ either has an element of order 16 or there are even number of terms $\mathbf{Z}_2$ and even number of terms $\mathbf{Z}_4$ in the canonical decomposition of G, moreover it is possible that the canonical decomposition of G contains only one term of the form $\mathbf{Z}_{2^t}$ for any $t \geq 3$. Denote by g the greatest common divisor of a and b. Since G is finite, we have $detA \neq 0$ and $g > 0$. It is easy to see that G can be presented by the diagonal matrix $\widetilde{A} = \begin{pmatrix} g & 0 \\ 0 & \frac{|detA|}{g} \end{pmatrix}$. Therefore $G \simeq \mathbf{Z}_g \oplus \mathbf{Z}_{|detA|/g}$ (here by $\mathbf{Z}_1$ we mean the trivial group).

If a and b are both odd then g is odd and $\frac{|detA|}{g}$ is divisible by 8.

If a and b are both even then g is even. Let t be the power of 2 in the prime decomposition of g. We have two different cases: 1) $\frac{|detA|}{g^2}$ is odd. Then $G \simeq \mathbf{Z}_{g/2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2l+1}$, where $2l + 1 = \frac{|detA|}{g2^t}$. 2) $\frac{|detA|}{g^2}$ is even. This means that both $\frac{a}{g}$ and $\frac{b}{g}$ are odd and therefore $\frac{|detA|}{g^2}$ is divisible by $2^s, s \geq 3$, which implies that $G \simeq \mathbf{Z}_{g/2^t} \oplus \mathbf{Z}_{2^t} \oplus \mathbf{Z}_{2^{s+t}} \oplus \mathbf{Z}_{2h+1}$, where $2h + 1 = \frac{|detA|}{g2^{s+t}}$. In this case we have an element of order 2^{s+t}, which is at least 16.

We are left with the case when one of the numbers a and b is even and the other one is odd. But in this case g is odd and $|detA|$ is odd.

Therefore, if G does not have elements of order 16 then the number of terms $\mathbf{Z}_2$ and the number of terms $\mathbf{Z}_4$ in the canonical decomposition of G are both even numbers. To see that it is possible to have a single $\mathbf{Z}_8$ term, consider the case $a = 3$ and $b = 1$ (An example of a manifold with such first homology is the lens space $L(8, 3)$). □

COROLLARY 2.21. *Let M be a rational homology 3-sphere such that the group $H_1(M; \mathbf{Z})$ does not have elements of order 8. If M admits an orientation preserving action of $\mathbf{Z}_2$ with the fixed point set being a circle then $H_1(M; \mathbf{Z}_2) \simeq \mathbf{Z}_2^m$ for some even integer m.*

CHAPTER 3
MURAKAMI-OHTSUKI-OKADA INVARIANTS AND DIJKGRAAF-WITTEN INVARIANTS AND PERIODIC 3-MANIFOLDS

In this section we will study two simple quantum invariants: Murakami-Ohtsuki-Okada invariants and Dijkgraaf-Witten invariants. We will concentrate on the values of these invariants for certain classes of periodic 3-manifolds. More specifically, we will relate values of the invariants for periodic 3-manifolds and their corresponding orbit spaces.

In Section 1 we give a definition and study some basic properties of Murakami-Ohtsuki-Okada invariants. In Section 2 we study Murakami-Ohtsuki-Okada invariants of p-periodic 3-manifolds, i.e., 3-manifolds with an action of $\mathbf{Z}_p$ such that the fixed point set is S^1. In Section 3 we define Dijkgraaf-Witten invariants. In Section 4 we study Dijkgraaf-Witten invariants of arbitrary 3-manifolds admitting $\mathbf{Z}_p$-actions.

Similar results for Reshetikhin-Turaev invariants and for Turaev-Viro invariants were obtained by P. Gilmer [Gi]. This part of our research was influenced by methods from [Gi] and [KB-Pr]. The results of this chapter were reported at the AMS meeting in Las-Vegas, Special Session on Symmetries of Knots and 3-Manifolds, April 1999 [S9].

3.1 Murakami-Ohtsuki-Okada invariants

Let $A = (\lambda_{ij})$ $(1 \leq i, j \leq n)$ be the linking matrix of an n-component oriented framed link L, that is, λ_{ij} is the linking number between i-th and j-th components of L and λ_{ii} is the framing number of the i-th component of L. Denote by $\sigma(A)$ the signature of A (the number of positive eigenvalues minus the number of negative eigenvalues). Let $N \geq 2$ and $d \geq 1$ be coprime integers with $N + d$ being odd, and put $q = e^{\frac{d\pi\sqrt{-1}}{N}}$. Note that q is a primitive N-th root of unity if N is odd and a primitive $2N$-th root of unity if N is even. The N-th Murakami-Ohtsuki-Okada invariant was defined in [MOO] by the following formula:

$$MOO_N(M_L; q) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l},$$

where M_L is the 3-manifold obtained by the surgery on L and $G_N(q) = \sum_{h \in \mathbf{Z}_N} q^{h^2}$. To simplify notations, sometimes we will write $MOO_N(M_L)$ instead of $MOO_N(M_L; q)$.

Notice, that $G_N(q)$ is the famous Gaussian sum. Its properties are well-known (see, for instance, [L]):

$$\sum_{h\in\mathbf{Z}_n} e^{\frac{2\pi i}{n}h^2} = \begin{cases} (1+i)\sqrt{n}, & \text{when } n=0 \pmod 4 \\ \sqrt{n}, & \text{when } n=1 \pmod 4 \\ 0, & \text{when } n=2 \pmod 4 \\ i\sqrt{n}, & \text{when } n=3 \pmod 4 \end{cases} \tag{3.1}$$

If n is odd and k is coprime to n then

$$\sum_{h\in\mathbf{Z}_n} e^{\frac{2\pi ik}{n}h^2} = \left(\frac{k}{n}\right)\sum_{h\in\mathbf{Z}_n} e^{\frac{2\pi i}{n}h^2}, \tag{3.2}$$

where $\left(\frac{k}{n}\right)$ is Legendre's symbol, that is $\left(\frac{k}{n}\right)=1$ if there exists an integer l such that $l^2=k \pmod n$, and $\left(\frac{k}{n}\right)=-1$ otherwise. If n is even and k is coprime to n then

$$\sum_{h\in\mathbf{Z}_n} e^{\frac{2\pi ik}{n}h^2} = 0 \Longleftrightarrow n=2 \pmod 4.$$

In particular, it follows that $G_N(q)\neq 0$.

By definition the values of $MOO_N(M;q)$ lie in the field $\mathbf{Q}(q)$. The following proposition easily follows from the equation (3.1).

PROPOSITION 3.1. *Consider the natural field monomorphism* $\eta\colon \mathbf{Q}(q)\to\mathbf{C}$ *defined by*

$$\eta(q) = \begin{cases} e^{2\pi i/N}, & \textit{if } N \textit{ is odd,} \\ e^{2\pi i/2N}, & \textit{if } N \textit{ is even.} \end{cases}$$

Then

$$\eta(G_N(q)) = \begin{cases} \sqrt{N}, & \textit{if } N\equiv 1 \textit{ mod } 4, \\ i\sqrt{N}, & \textit{if } N\equiv 3 \textit{ mod } 4, \\ (1+i)\sqrt{N/2}, & \textit{if } N \textit{ is even.} \end{cases}$$

In particular, $|G_N(q)|^2 = G_N(q)\cdot\overline{G_N(q)} = N$.

PROOF. If N is odd then the proposition is obvious. Assume that N is even. In this case

$$\eta(G_N(q)) = \sum_{h\in\mathbf{Z}_N}(e^{2\pi i/2N})^{h^2} = \frac{1}{2}\sum_{h\in\mathbf{Z}_{2N}} e^{2\pi i h^2/2N} = \frac{1}{2}(1+i)\sqrt{2N} = (1+i)\sqrt{\frac{N}{2}}.$$

To finish the proof, notice that if $\alpha\in\mathbf{Q}$ then $\eta(\alpha)=\alpha$. $\square$

The following useful theorem was proven in [MOO]. Here $\cup$ means the usual cup product.

THEOREM 3.2. ([MOO]) *If there exists* $\alpha\in H^1(M;\mathbf{Z}_N)$ *with* $\alpha\cup\alpha\cup\alpha\neq 0$ *then*

$MOO_N(M) = 0$, otherwise $|MOO_N(M)| = |H^1(M;\mathbf{Z}_N)|^{1/2}$, where $|\cdot|$ in the right hand side is the order of the set. For odd N, $\alpha\cup\alpha\cup\alpha$ is always zero.

In order to simplify the statements of our results we will sometimes use a different normalization for Murakami-Ohtsuki-Okada invariants:

$$MOO_N^*(M;q) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\beta_1(M)} \cdot MOO_N(M;q),$$

where $\beta_1(M)$ is the first Betti number of M. Notice that with notations as before,

$$MOO_N^*(M_L;q) = (G_N(q))^{-b_+-\beta_1(M_L)} \cdot (\overline{G_N(q)})^{-b_-} \cdot \sum_{l\in\mathbf{Z}_N^n} q^{l^tAl}, \tag{3.3}$$

where b_+ and b_- are the numbers of positive and negative eigenvalues of A. Indeed,

$$(G_N(q))^{-b_+-\beta_1(M)} \cdot (\overline{G_N(q)})^{-b_-} = \frac{(G_N(q))^{-\sigma}}{|G_N(q)|^{2b_-}} \cdot (G_N(q))^{-\beta_1(M)} =$$
$$= \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma-\beta_1(M)} \cdot |G_N(q)|^{-n}.$$

The last equation follows from the fact that $\beta_1(M)$ is equal to the nullity of A.

Lemma 3.3.

$$MOO_N^*(M;q) \in \mathbf{Z}[q,1/N]$$

Proof. This follows from the equation (3.3) and the fact that $G_N(q)\overline{G_N(q)} = N$ (see Proposition 3.1). □

The following theorem will be very important in the next section. Here $rkH_1(M;\mathbf{Z}_p)$ means the rank of $H_1(M;\mathbf{Z}_p)$ as a vector space over $\mathbf{Z}_p$.

Theorem 3.4. *Let p be an odd prime integer. Then*

$$MOO_p^*(M;q) = \pm(G_p(q))^{rkH_1(M;\mathbf{Z}_p)}.$$

In particular, $MOO_p^*(M;q) \in \mathbf{Z}[q]$.

Proof. Let L be an n component framed link, surgery on which produces M. Let A be the linking matrix of L. Since p is odd, we can diagonalize A as a matrix over $\mathbf{Z}_p$, that is, there exists a matrix $S \in SL(n,\mathbf{Z})$ such that

$$S^tAS = \bigoplus_{j=1}^n (d_j) \pmod p.$$

(see, for instance, [MOO], p. 554)

We can assume that $d_1, \ldots, d_k$ are not zeros modulo p and $d_{k+1} = \ldots = d_n = 0 \pmod p$. Notice that $G_p(1) = p$, therefore

$$\sum_{l \in \mathbf{Z}_p^n} q^{l^t A l} = \prod_{i=1}^{n} \left(\sum_{h \in \mathbf{Z}_p} q^{d_i h^2} \right) = \prod_{i=1}^{n} G_p(q^{d_i}) = p^{n-k} \prod_{i=1}^{k} G_p(q^{d_i}).$$

Equation (3.2) implies that for $1 \leq i \leq k$, $G_p(q^{d_i}) = \pm G_p(q)$. Thus,

$$\sum_{l \in \mathbf{Z}_p^n} q^{l^t A l} = \pm p^{n-k} (G_p(q))^k.$$

Since p is odd, $\overline{G_p(q)} = \pm G_p(q)$, and $p = \pm G_p(q)^2$ (see Proposition 3.1). Therefore, by equation (3.3) we have

$$MOO_p^*(M; q) = \pm (G_p(q))^{-b_+ - (n-k) - b_- + k} \cdot p^{n-k} = \pm (G_p(q))^{n-k}.$$

It remains to notice that $n - k = rkH_1(M; \mathbf{Z}_p)$. □

3.2 Murakami-Ohtsuki-Okada invariants and periodic 3-manifolds

Fix $N \geq 2$ and let q be a root of unity as before. Let L be an n component framed link with the linking matrix A. We will use the following notation:

$$[L]_q = \sum_{l \in \mathbf{Z}_p^n} q^{l^t A l}.$$

Notice that $[L]_q \in \mathbf{Z}[q]$.

Throughout the rest of this chapter we will use the following convention. For $a, b \in \mathbf{Z}[q]$ we will write $a = b \pmod p$ meaning that a and b are equal as polynomials in q with coefficients treated modulo p. If $a, b \in \mathbf{Z}[q, 1/N]$ where N is coprime to p, then N can be inverted modulo p, and therefore modulo p, a and b can be considered as elements of $\mathbf{Z}_p[q]$, and the equation $a = b \pmod p$ in this case will mean that a and b are equal as elements of $\mathbf{Z}_p[q]$.

Lemma 3.5. *Let L^p be a strongly p-periodic framed link for some prime integer p. Let L_* be its underlying link (see Chapter 2 for definitions). Then*

$$[L^p]_q = ([L_*]_q)^p \pmod p.$$

Proof. Notice that the vectors $l \in \mathbf{Z}_p$ in the above sum can be treated as colorings of components of L by the elements of $\mathbf{Z}_N$. Every such coloring of L_* can be lifted to a coloring of L^p such that the corresponding terms in the sums $[L_*]_q$ and $[L^p]_q$ are related as q^x and q^{px}. All other colorings of L^p are permuted by the $\mathbf{Z}_p$ action, therefore their contribution to the sum is 0 modulo p. Thus, if $[L_*]_q = q^{x_1} + \ldots + q^{x_k}$, then $[L^p]_q = q^{px_1} + \ldots + q^{px_k} = (q^{x_1} + \ldots + q^{x_k})^p \pmod p$. □

THEOREM 3.6. *Let M be a closed oriented 3-manifold and $\mathbf{Z}_p$ acts on M with the fixed point set S^1 for some prime p. Let M_* be the orbit space of the action. If N is an odd integer coprime to p, then*

$$MOO^*_N(M;q) = \begin{cases} (MOO^*_N(M_*;q))^p \pmod p & \text{if } N \equiv 1 \text{ mod } 4, \\ \pm(MOO^*_N(M_*;q))^p \pmod p & \text{if } N \equiv 3 \text{ mod } 4. \end{cases}$$

PROOF. By Theorem 2.1, M can be obtained by the surgery on a strongly p-periodic framed link L^p. Let L_* be the underlying link. The manifold M_* can be obtained by surgery on L_*.

Case $N \equiv 1 \pmod 4$. In this case $\overline{G_p(q)} = G_p(q)$ (see Proposition 3.1), therefore

$$\frac{1}{G_N(q)} = \frac{G_N(q)}{N}.$$

$$MOO^*_N(M_*;q) = (G_N(q))^{-n}[L_*]_q = \frac{(G_N(q))^n}{N^n} \cdot [L_*]_q,$$

where n is the number of components of L_*. By Lemma 2.5, the number of components of L^p is pn. Therefore,

$$MOO^*_N(M;q) = \frac{(G_N(q))^{pn}}{N^{pn}} \cdot [L^p]_q.$$

Let $\tilde{N}$ be an integer such that $\tilde{N}N \equiv 1 \pmod p$. Then

$$MOO^*_N(M_*;q) = \tilde{N}^n (G_N(q))^n [L_*]_q \in \mathbf{Z}_p[q]$$

and

$$MOO^*_N(M;q) = \tilde{N}^{np} (G_N(q))^{np} ([L_*]_q)^p \in \mathbf{Z}_p[q],$$

where the last equation follows from Lemma 3.5.

Case $N \equiv 3 \pmod 4$. By Proposition 3.1, $\overline{G_p(q)} = -G_p(q)$. Repeat the same steps as in the previous case. Unfortunately, we do not have much control over the sign, therefore we leave $\pm$ in the formula. □

REMARK 3.7. Note that Lemma 3.5 does not give criterion for periodicity of links. For any framed link L there is a strongly p-periodic framed link L^p such that $[L]_q = [L^p]_q \pmod p$ as long as $(N,p) = 1$. Indeed, assume $[L]_q = a_0 + a_1 q + \ldots + a_k q^k \in \mathbf{Z}[q]$ and let A be the linking matrix of L. Let L_* be a framed link with the linking matrix

$\tilde{p}A$, where $\tilde{p}$ is the inverse of p in $\mathbf{Z}_N$. Let L^p be a strongly p-periodic framed link such that L_* is its underlying link. By lemma 3.5, $[L]_q = [L^p]_q \pmod p$.

The previous remark proves that Theorem 3.6 does not provide any restrictions for 3-manifolds to be p-periodic. A non-trivial periodicity criterion is given by the following theorem.

THEOREM 3.8. *Suppose that a closed orientable 3-manifold M admits a $\mathbf{Z}_p$ action with the fixed point set equal S^1 for some odd prime p. Then either $MOO_p^*(M;q) = \pm 1$, or $MOO_p^*(M;q) = 0 \pmod p$.*

PROOF. First of all, it makes sense to consider $MOO_p^*(M;q)$ modulo p, because by Theorem 3.4, $MOO_p^*(M;q) \in \mathbf{Z}[q]$. Moreover, by the same theorem, $MOO_p^*(M;q) = \pm(G_p(q))^{rkH_1(M;\mathbf{Z}_p)}$. By Theorem 2.3, $rkH_1(M;\mathbf{Z}_p) \neq 1$. Since $(G_p(q))^2 = \pm p$ (see Proposition 3.1), for any $k \geq 2$ we have $(G_p(q))^k = 0 \pmod p$. The result follows. □

3.3 Dijkgraaf-Witten invariants

Dijkgraaf-Witten invariants of 3-manifolds were defined by Dijkgraaf and Witten in [DW] in 1990. Combinatorial definitions were given later in several papers, see for instance [Wak] and [FG]. Before we define Dijkgraaf-Witten invariants we need to recall some basic group cohomology. We will mostly follow [Wak]. See, also, [Br] and [AM] for further details. Let G be a finite group and V be a multiplicative abelian group. Denote the set of all maps

$$f \colon \underbrace{G \times \ldots \times G}_{n} \to V$$

by $C^n(G;V)$. Define a coboundary operator $\delta^n \colon C^n(G;V) \to C^{n+1}(G;V)$ by the following formula:

$$\begin{aligned}(\delta^n f)(x_1, \ldots, x_{n+1}) &= f(x_2, \ldots, x_{n+1}) \cdot \\ &\cdot \left(\prod_{i=1}^{n} f(x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \right) f(x_1, \ldots, x_n)^{(-1)^{n+1}}\end{aligned}$$

In the usual way, set

$$H^n(G;V) = \text{Ker } \delta^n / \text{Im } \delta^{n-1}.$$

Fix a finite group G and $\alpha \in H^3(G;\mathbf{Q}/\mathbf{Z})$. Let M be a compact oriented 3-manifold. Fix a triangulation T of M. Denote (and thus order) the vertices of the triangulation by $v_1, \ldots, v_n$. By a coloring of T we mean a map

$$\phi \colon \{\text{edges of } T\} \to G.$$

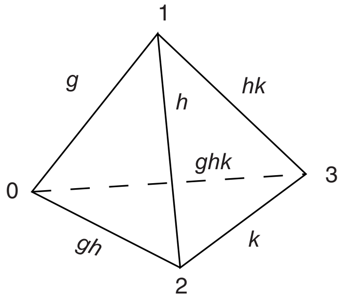


Fig. 3.1.

A coloring ϕ is called *admissible* ($\phi \in Adm_G T$) if for each face c of the triangulation we have $\phi(e_{01})\phi(e_{12})\phi(e_{20})^{-1} = 1$ in G. Here e_{01}, e_{12}, and e_{20} are the edges of c connecting the vertices v_i and v_j, v_j and v_k, and v_k and v_i respectively, where $i < j < k$.

Let t be a tetrahedron of T with the vertices $v_{i(0)}, v_{i(1)}, v_{i(2)}, v_{i(3)}$, where $i(0) < i(1) < i(2) < i(3)$. Denote the edges of t by e_{ks}, $k, s = 0, 1, 2, 3$, such that e_{ks} is the edge connecting the vertices $v_{i(k)}$ and $v_{i(s)}$. Notice, that any admissible coloring ϕ of t is completely defined by $\phi(e_{01})$, $\phi(e_{12})$, and $\phi(e_{23})$ (see Fig. 3.1).

Notice that the ordering of the vertices of t induces an orientation on t. Set

$$\varepsilon(t) = \begin{cases} 1, & \text{if the orientation of } t \text{ induced by the ordering of the vertices} \\ & \text{and by the orientation of the manifold coincide;} \\ -1, & \text{otherwise.} \end{cases}$$

Fix

$$W(t, \phi) = \alpha(\phi(e_{01}), \phi(e_{12}), \phi(e_{23}))^{\varepsilon(t)} \in \mathbf{Q}/\mathbf{Z}.$$

Dijkgraaf-Witten invariant for the 3-manifold M, finite group G and $\alpha \in H^3(G; \mathbf{Q}/\mathbf{Z})$ is defined as follows

$$DW_{G,\alpha}(M) = \frac{1}{|G|^{n-1}} \sum_{\phi \in Adm_G T} \prod_{t \in T} W(t, \phi)$$

(compare to the definition of Turaev-Viro invariants from Chapter 1), where $|G|$ denotes the order of the group G.

Remark 3.9. Since G is finite, for any fixed $\alpha \in H^3(G; \mathbf{Q}/\mathbf{Z})$ we have Im $\alpha \subset < q >$, where $< q >$ is the cyclic group generated by a certain root of unity q. Thus, by definition, $DW_{G,\alpha}(M) \in \mathbf{Z}[\frac{1}{|G|}, q]$.

Remark 3.10. Our normalization is chosen so that

$$DW_{G,\alpha}(M) = |G| \cdot \ \{\text{invariant defined in [Wak]}\}.$$

The benefit of our normalization is that if α is trivial and M is closed then

$$DW_{G,\alpha}(M) = \{\text{number of representations of } \pi_1(M) \text{ over } G\}$$

(see, for instance, Remark on p. 679 from [Wak]).

REMARK 3.11. Normally, Dijkgraaf-Witten invariants are defined using $\mathbf{R}/\mathbf{Z}$ in the places where we use $\mathbf{Q}/\mathbf{Z}$. Our restriction for the coefficients of the group cohomology allows the important property $DW_{G,\alpha}(M) \in \mathbf{Z}[\frac{1}{|G|}, q]$ (see Remark 3.9). In fact, since G is finite, we suspect that using $\mathbf{Q}/\mathbf{Z}$ instead of $\mathbf{R}/\mathbf{Z}$ does not change the result anyway, but we decided to leave further investigation of this question to the readers. We only want to add that since $\mathbf{Q}/\mathbf{Z} = U(1, \mathbf{Q})$, our choice for the coefficients is entirely valid (see [FG]).

REMARK 3.12. It follows from Remark 3.10 that if G is a finite abelian group, α is the trivial cocycle, and M is a closed oriented 3-manifold, then

$$DW_{G,\alpha}(M) = |H^1(M; G)|.$$

In particular, by Theorem 3.2, if α is trivial and for each $\beta \in H^1(M; \mathbf{Z}_N)$ we have $\beta \cup \beta \cup \beta = 0$, then

$$DW_{\mathbf{Z}_N,\alpha}(M) = |MOO_N(M)|^2.$$

In Appendix A we will show how to define Dijkgraaf-Witten invariants using singular triangulations and simple and special spines of 3-manifolds.

3.4 Dijkgraaf-Witten invariants and periodic 3-manifolds

In this section we will follow the same conventions for equivalence modulo p in $\mathbf{Z}[\frac{1}{|G|}, q]$ as in section 2.

THEOREM 3.13. *Let p be a prime number and M be an oriented 3-manifold such that $\mathbf{Z}_p$ acts on M. Let M_* be the orbit space of the action, that is $M_* = M/\mathbf{Z}_p$. For any finite group G whose order is not divisible by p and for any $\alpha \in H^3(G; \mathbf{Q}/\mathbf{Z})$ we have*

$$DW_{G,\alpha}(M) = (DW_{G,\alpha}(M_*))^p \pmod{p}.$$

PROOF. Let $\gamma \in M$ be the fixed point set of the action, and let $\gamma_* \in M_*$ be the image of γ under the natural map $M \to M_* = M/\mathbf{Z}_p$. Fix a triangulation T_* of M_* such that γ_* lies in the 1-dimensional subset of T_*. Let T be the triangulation of M lifted from T_*. Order the vertices of T_*. This ordering induces an orientation of the edges of T_*, and thus, an orientation of the edges of T. Note, that the orientation of the edges defines an ordering of the vertices for each tetrahedron of T, thus it provides sufficient data for calculation of the invariant. It follows from our construction that each admissible coloring of T_* lifts to an admissible coloring of T. As each tetrahedron of T_* is covered by p tetrahedra of T, the contribution of a lifted coloring to the sum for $DW_{G,\alpha}(M)$ is the p-th power of the contribution of the original coloring to $DW_{G,\alpha}(M_*)$. $\mathbf{Z}_p$ acts freely on the set of colorings of T which are not lifts of some coloring of T_*. Thus, the contribution of the non-equivalent colorings is a multiple of p. Making use of the equation $x^p + y^p = (x+y)^p \pmod p$ we have

$$\sum_{\phi \in Adm_G T} \prod_{t \in T} W(t,\phi) = \left(\sum_{\phi \in Adm_G T_*} \prod_{t \in T_*} W(t,\phi) \right)^p \pmod p$$

Let n be the number of vertices of T and n_* be the number of vertices of T_*. Let h be the number of vertices of T_* that lie on γ_*. Set $k = n_* - h$. Then $n = h + pk$. Since $|G|^{hp} = |G|^h \pmod p$, we have

$$|G|^{n_* p} = |G|^{hp} |G|^{kp} = |G|^n \pmod p.$$

Therefore,

$$\frac{1}{|G|^{(n_*-1)p}} = \frac{1}{|G|^{n-1}} \pmod p.$$

The result follows. □

COROLLARY 3.14. *If G is abelian, then $|H^1(M;G)| = |H^1(M_*;G)| \pmod p$.*

THEOREM 3.15. *Suppose that a closed orientable 3-manifold M admits a $\mathbf{Z}_p$ action with the fixed point set equal to S^1 for some odd prime p. Let α be the trivial element of $H^3(\mathbf{Z}_p; \mathbf{Q}/\mathbf{Z})$. Then $DW_{\mathbf{Z}_p,\alpha}(M) \neq p$.*

PROOF. Directly follows from Remark 3.12 and Theorem 2.3. □

CHAPTER 4
TWO-TERM SKEIN RELATIONS AND QUANTUM INVARIANTS OF THREE-MANIFOLDS

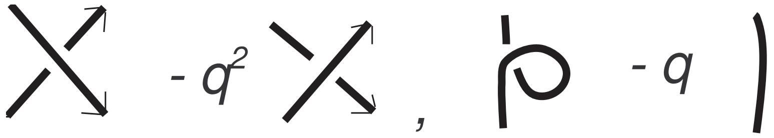
In 1990 Lickorish showed how to construct Reshetikhin-Turaev ($SU(2)$) invariants from the Kauffman bracket skein module (see [Li2], [Li3], [Li4]). Recently, Y. Yokota constructed other quantum $SU(N)$-invariants using Lickorish method on HOMFLY skein module [Yo]. For an overview and some generalizations of Lickorish method see [B]. In section 1 we will remind definitions of the simplest known skein modules, $\mathcal{S}$ and $\mathcal{S}_2$ (the later is also known as the q-analog of the first homology, see [Pr1], [Pr4]), and Lickorish's method for constructing 3-manifold invariants. In section 2 we will show how to construct Murakami-Ohtsuki-Okada invariants ([MOO]) using Lickorish's method. In section 3 we will show that if q is a root of unity then no other invariants can be obtained using Lickorish's method on $\mathcal{S}$ and $\mathcal{S}_2$ (see Theorem 4.7).

In Section 4 we study Deloup's invariants $\tau(M,G,\xi)$ (see [De]). Essentially introduced by V. Turaev via a simple example of modular categories [T2], these invariants are an interesting generalization of the Murakami-Ohtsuki-Okada invariants for 3-manifolds. Here G is a finite abelian group and $\xi : G \to \mathbf{Q}/\mathbf{Z}$ is a quadratic form. Deloup showed that these invariants depend only on the first Betti number and the linking form of the manifold. Recently, Deloup and Gille proved that these invariants keep all the information about the linking form and the first Betti number ([DG]). Moreover, they proved that all the information can, in fact, be recovered from two special cases: $\tau(M, \mathbf{Z}_{p^n}, (1/p^n))$, for prime p, and $\tau(M, \mathbf{Z}_{2^n} \times \mathbf{Z}_{2^n}, Q_n)$. Here $\tau(M, \mathbf{Z}_{p^n}, (1/p^n))$ coincides with the Murakami-Ohtsuki-Okada invariant of order p^n, and Q_n is the quadratic form defined by $Q_n(x,y) = xy/2^n$. As we mentioned above, Murakami-Ohtsuki-Okada invariants can be obtained using Lickorish construction on the skein module $\mathcal{S}_2$ (q-analogue of the first homology). In section 4, we show that $\tau(M, \mathbf{Z}_{2^n} \times \mathbf{Z}_{2^n}, Q_n)$ can be similarly defined using another simple skein module, we call it *dichromatic skein module*, which turned out to be the tensor product of two copies of $\mathcal{S}_2$ (see section 5). Using the dichromatic skein module we give a simple definition of a linking number between two links in an oriented 3-manifold (compare to [Ka]).

The results from sections 1, 2 and 3 where presented at the conference *Knots in Hellas 98* [S10]. After the talk the author was told that for the skein module $\mathcal{S}_2$ very similar results where obtained by Mohamed Ferhane [Fe]. Most of the results from sections 4 and 5 were presented at the *Joint Mathematics Meeting* in Washington, DC in January 2000, Special Session on Invariants of Knots and 3-Manifolds ([S8]).

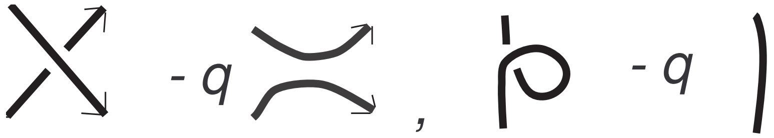
4.1 Skein modules $\mathcal{S}$ and $\mathcal{S}_2$ and Lickorish method for constructing 3-manifold invariants

Definition. ([Pr1]) Let $\mathcal{L}^f$ denote the set of framed oriented links in an oriented 3-manifold M (including the empty link $\emptyset$). Let R be a commutative link with identity. Let us consider the submodule S^f of $R\mathcal{L}^f$ which is generated by the following skein expressions:



We define the skein module $\mathcal{S}$ as $\mathcal{S}(M; R, q) = R\mathcal{L}^f/S^f$. We will denote $\mathcal{S}(M; \mathbf{Z}[q^{\pm 1}], q)$ by $\mathcal{S}(M; q)$.

Definition. ([Pr1], [Pr4]) Let $\mathcal{L}^f_* = \mathcal{L}^f - \{\emptyset\}$, where $\emptyset$ denotes the empty link. Let S^f_2 be the submodule of $R\mathcal{L}^f_*$ generated by the following skein expressions:



Define the skein module $\mathcal{S}_2$ (q-analog of the first homology group) by $\mathcal{S}_2(M; R, q) = R\mathcal{L}^f_*/S^f_2$. We will denote $\mathcal{S}_2(M; \mathbf{Z}[q^{\pm 1}], q)$ by $\mathcal{S}_2(M; q)$.

Remark 4.1 The skein modules $\mathcal{S}(S^3; q)$ and $\mathcal{S}_2(S^3; q)$ have a natural algebra structure with multiplication defined by the disjoint union of links.

Let $T_n \subset S^3$ denote the trivial framed link of n components.

Proposition 4.2. *1) $\mathcal{S}(S^3; q) \simeq \mathbf{Z}[q^{\pm 1}, \alpha]$ as algebras, with isomorphism defined by*

$$\varphi: T_n \mapsto \alpha^n, \quad n \geq 0.$$

2) For any link $L \subset S^3$, $\varphi(L) = q^x \alpha^n$, where x is the sum of all elements of the linking matrix of L and n is the number of components of L.
3) $\mathcal{S}_2(S^3; q) \simeq \mathbf{Z}[q^{\pm 1}]$ as algebras, with isomorphism defined by

$$\varphi_2: T_n \mapsto 1, \quad n \geq 1.$$

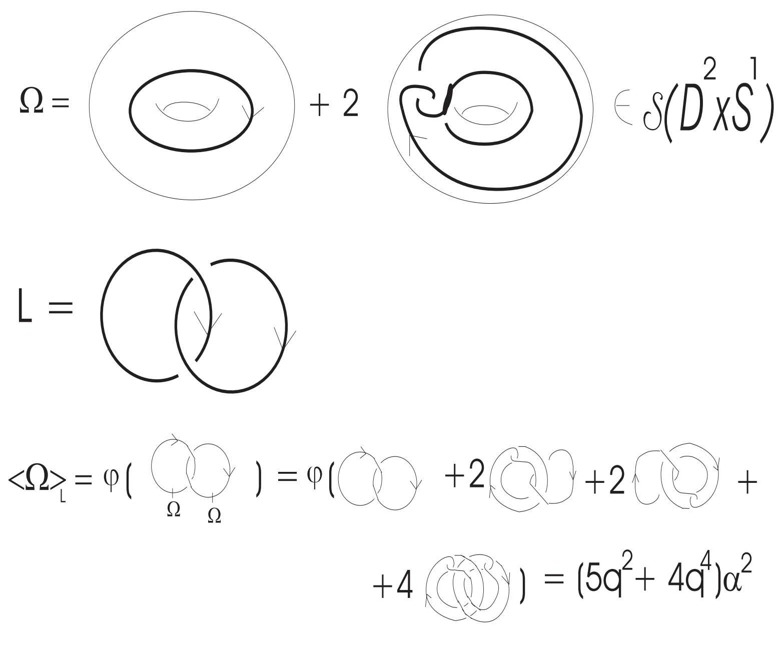
4) The algebras $\mathcal{S}(S^3; q)/(T_1 - \emptyset)$ and $\mathcal{S}_2(S^3; q)$ are isomorphic.
Proof.
2) Obvious.
1) Consider a map $\psi: \mathbf{Z}[q^{\pm 1}, \alpha] \to \mathcal{S}(S^3; q)$ defined by $\psi(q^x \alpha^n) = q^x T_n$. It is an algebra homomorphism. Moreover, φ and ψ are inverses of each other. Indeed, the

part $\varphi \circ \psi = \mathrm{id}$ is trivial. The equation $\psi \circ \varphi = \mathrm{id}$ follows from the fact that in $\mathcal{S}(S^3; q)$ any link L represents the same element as $q^x T_n$ iff x is equal to the sum of all elements of the linking matrix of L and n is the number of components of L. Thus, φ is one-to-one, hence, it is an algebra isomorphism.
3) The proof is similar to the proof of part 1. Instead of φ consider φ_2. Notice that for a link $L \subset S^3$, $\varphi_2(L) = q^x$, with notations as before. The inverse map can be defined by $\psi_2(q^x) = q^x T_1$.
4) An isomorphism is given by $\psi_2 \circ (\varphi_{|_{\mathcal{S}(S^3;q)/(T_1-\emptyset)}})$ with $\psi \circ \varphi_2$ as the inverse. $\square$

Fix an element $\Omega \in \mathcal{S}(D^2 \times S^1; q)$ (or $\mathcal{S}_2(D^2 \times S^1; q)$). Any framed oriented link $L \subset S^3$ with n components defines an oriented embedding of disjoint union of n solid tori $D^2 \times S^1$ into S^3, thus defines a multilinear map $\xi^L: \mathcal{S}(D^2 \times S^1; q)^n \to \mathcal{S}(S^3; q)$ (or $\xi_2^L: \mathcal{S}_2(D^2 \times S^1; q)^n \to \mathcal{S}_2(S^3; q)$). By $< \Omega >_L$ we will denote the element of $\mathbf{Z}[q^{\pm 1}, \alpha]$ (or $\mathbf{Z}[q^{\pm 1}]$) equal to $\varphi(\xi^L(\Omega \times \ldots \times \Omega))$ (or $\varphi_2(\xi_2^L(\Omega \times \ldots \times \Omega))$), where φ (or φ_2) is the isomorphism defined in Proposition 4.2.

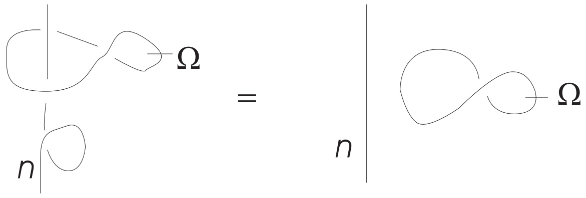
We will adopt the following traditional notations for our diagrams: an integer n next to a string means n parallel copies of that string (with regard to the plane of the diagram), an element $\Omega \in \mathcal{S}(D^2 \times S^1; q)$ (or $\mathcal{S}_2(D^2 \times S^1; q)$) next to a knot diagram K means the skein element $\xi^K(\Omega)$ (or $\xi_2^K(\Omega)$), i.e. the embedding of Ω into a neighborhood of K specified by the framing of K.

Example.



The following proposition easily follows from Kirby's theorem [K] as refined in [FR]. This method for construction of 3-manifold invariants will be called *Lickorish's method.*

PROPOSITION 4.3. *Assume, there are complex substitutions for q and α and there is an element $\Omega \in \mathcal{S}(D^2 \times S^1; q)$ (or $\Omega \in \mathcal{S}_2(D^2 \times S^1; q)$) such that the following conditions hold:*
1) For any framed link L, $<\Omega>_L$ does not depend on orientation of L;
2) Let α^+ and α^- be the trivial knots with framings 1 and -1 correspondingly, then $<\Omega>_{\alpha^+}$ and $<\Omega>_{\alpha^-}$ are not zeros;
3) The following equation holds in $\mathcal{S}(S^3; q)$ (or $\mathcal{S}_2(S^3; q)$) for any $n > 0$:



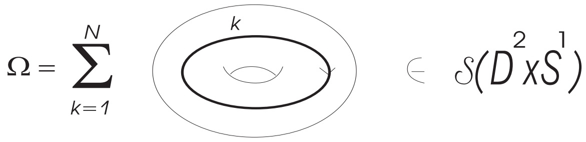
then the number

$$Inv_\Omega(M_L) = \frac{<\Omega>_L}{<\Omega>_{\alpha^+}^{b_+} <\Omega>_{\alpha^-}^{b_-}}$$

is an invariant of the 3-manifold M_L obtained by Dehn surgery on the framed link L. Here b_+ and b_- are the numbers of positive and negative eigenvalues of the linking matrix of L. □

4.2 Murakami-Ohtsuki-Okada invariants from Lickorish's method

THEOREM 4.4 (Main Example) *Fix $N > 2$ such that $N \neq 2 \pmod 4$, and*



Let $\alpha = 1$, and q be a primitive N-th root of unity. Then the function

$$Inv_\Omega(M_L) = Inv_\Omega(M_L; q) = \frac{<\Omega>_L}{<\Omega>_{\alpha^+}^{b_+} <\Omega>_{\alpha^-}^{b_-}}$$

is a non-trivial invariant of 3-manifolds.

PROOF. We only need to check that our choice for N, q, and Ω satisfies the conditions of Lickorish method as stated in Proposition 4.3. Indeed, condition (3) is equivalent to the equation

$$\sum_{k \in \mathbf{Z}_N} q^{(n+k)^2} = \sum_{k \in \mathbf{Z}_N} q^{k^2}, \ n = 1, 2, \ldots .$$

This holds because k and $(n+k)$ modulo N run over the same set of values. Note that $<\Omega>_{\alpha^+} = \overline{<\Omega>_{\alpha^-}} = \sum_{k\in \mathbf{Z}_N} q^{k^2} = 0$ iff $N = 2 \pmod 4$ (see [L]), thus we have the condition (2).

Let framed links L and L' be different only by the orientation of the last component l_n, then $<\Omega>_L = <\Omega>_{L'}$. Indeed, with notations as before, $<\Omega>_L = \varphi_2(\xi_L(\Omega,\dots,\Omega))$, where $\xi_L(\Omega,\dots,\Omega)$ is a sum of links in S^3, where each term is obtained by taking k_i parallel copies of the component l_i of L, with $(k_1,\dots,k_n)$ running over $\mathbf{Z}_N^n$. In our skein module, the effect of changing the orientation for l_n is the same as after changing each vector $(k_1,\dots,k_n)$ to $(k_1,\dots,k_{n-1},N-k_n)$, which does not change the total set of the vectors. Thus, the condition (1) of Lickorish method also holds. Easy calculations on simple examples show that the obtained invariant is not trivial. □

REMARK 4.5. It follows from Proposition 4.2 (part 4) that constructing 3-manifold invariants using Lickorish method on the skein module $\mathcal{S}$ with $\alpha = 1$ is the same as using Lickorish method on the skein module $\mathcal{S}_2$. Therefore, the above theorem could be formulated with $\Omega \in \mathcal{S}_2(D^2\times S^1; q)$.

The next proposition shows that the invariants described in Theorem 4.4 essentially coincide with the Murakami-Ohtsuki-Okada invariants. Here $\beta_1(M_L)$ denotes the first Betti number of M_L. For definition and properties of Murakami-Ohtsuki-Okada invariants see Chapter 3.

PROPOSITION 4.6. *Let N be a positive integer such that $N > 2$ and $N \neq 2 \pmod 4$. Let q be a primitive N-th root of unity and $Inv_\Omega(M_L;q)$ be the 3-manifold invariant introduced in Theorem 4.4. Let L be an oriented framed link with n components and with linking matrix A. If N is odd then*

$$Inv_\Omega(M_L;q) = |G_N(q)|^{\beta_1(M_L)} MOO_N(M_L;q);$$

if N is even then

$$Inv_\Omega(M_L;q) = |2G_{\frac{N}{2}}(q)|^{\beta_1(M_L)} MOO_{\frac{N}{2}}(M_L;q).$$

PROOF. 1) Let N be odd. Then

$$<\Omega>_{\alpha^+} = \sum_{h\in\mathbf{Z}_N} q^{h^2} = G_N(q),$$

$$<\Omega>_{\alpha^-} = \sum_{h\in\mathbf{Z}_N} q^{-h^2} = \overline{G_N(q)}.$$

It follows from Proposition 4.2 that

$$< \Omega >_L = \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l}.$$

Therefore,

$$Inv_\Omega(M_L; q) = G_N(q)^{-b_+} (\overline{G_N(q)})^{-b_-} \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l} =$$

$$= \left(\frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} |G_N(q)|^{-n+\beta_1} \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l},$$

where β_1 is the nullity of A, that is the first Betti number of M_L.

2) Now consider the case when $N = 0 \pmod 4$. In this case

$$< \Omega >_{\alpha^+} = \sum_{h \in \mathbf{Z}_N} q^{h^2} = 2 \sum_{h \in \mathbf{Z}_{\frac{N}{2}}} q^{h^2} = 2G_{\frac{N}{2}}(q).$$

Similarly,

$$< \Omega >_{\alpha^-} = 2\overline{G_{\frac{N}{2}}(q)}, \qquad \text{and}$$

$$< \Omega >_L = \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l} = 2^n \sum_{l \in \mathbf{Z}_{\frac{N}{2}}^n} q^{l^t A l}.$$

Therefore,

$$Inv_\Omega(M_L; q) = 2^{n-b_+-b_-} G_{\frac{N}{2}}(q)^{-b_+} (\overline{G_{\frac{N}{2}}(q)})^{-b_-} \sum_{l \in \mathbf{Z}_{\frac{N}{2}}^n} q^{l^t A l} =$$

$$= 2^{\beta_1} \left(\frac{G_{\frac{N}{2}}(q)}{|G_{\frac{N}{2}}(q)|} \right)^{-\sigma(A)} |G_{\frac{N}{2}}(q)|^{-n+\beta_1(M_L)} \sum_{l \in \mathbf{Z}_{\frac{N}{2}}^n} q^{l^t A l}. \quad \square$$

4.3 Uniqueness theorem

The following theorem shows that if we will substitute a root of unity for q, then the only 3-manifold invariants that can be produced from the skein modules $\mathcal{S}$ and $\mathcal{S}_2$ using Lickorish method are Murakami-Ohtsuki-Okada invariants.

THEOREM 4.7. *Let q be a root of unity and let N be the smallest integer such that $q^N = 1$. Assume that Lickorish method applied to some $\Omega \in \mathcal{S}(D^2 \times S^1; q)$ produces a 3-manifold invariant $Inv_\Omega(M_L)$. Then*

1) $N \neq 2 \pmod 4$*;*
2) There is a constant C *such that*

$$Inv_\Omega(M_L) = C^{\beta_1(M_L)} MOO_K(M_L; q),$$

where $K = N$ *if* N *is odd and* $K = \frac{N}{2}$ *if* N *is even.*

PROOF. By definition, Ω can be represented as a finite linear combination of oriented framed links in $D^2 \times S^1$. Let $\omega_i \in \mathcal{S}(D^2 \times S^1)$ be the part of that linear combination, consisting of all the links for which the algebraic intersection number with the meridian disk D^2 is equal to i. Then $\Omega = \sum_{i=m}^{M} \omega_i$ for some integers m and M. Sometimes, it will be convenient to assume that $\Omega = \sum_{i \in \mathbf{Z}} \omega_i$ with the condition that the "extra" terms are of the form $0 \cdot L$. With this convention and Proposition 4.2 (part 2) in mind, for any n-component framed link L we have

$$< \Omega >_L = \sum_{l \in \mathbf{Z}^n} q^{l^t A l} \cdot \prod_{s=1}^{n} < \omega_{l_s} >_{T_1},$$

where l_s is the s-th component of the vector l, T_1 is the trivial knot, and A is the linking matrix of L. Notice, that if $< \omega_i >_{T_1} \neq 0$ and $< \omega_{i+tN} >_{T_1} \neq 0$ then

$$\frac{< \omega_{i+tN} >_L}{< \omega_{i+tN} >_{T_1}} = \frac{< \omega_i >_L}{< \omega_i >_{T_1}},$$

for any $t \in \mathbf{Z}$. This easily follows from the fact that $q^N = 1$. Therefore, with the notation

$$W_i = \sum_{t \in \mathbf{Z}} < \omega_{i+tN} >_{T_1}$$

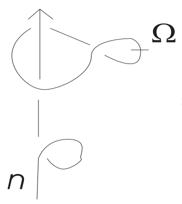
the above summation can be rewritten in the following form

$$< \Omega >_L = \sum_{l \in \mathbf{Z}_N^n} q^{l^t A l} \cdot \prod_{s=1}^{n} W_{l_s}.$$

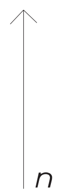
In particular,

$$< \Omega >_{\alpha^+} = \sum_{h \in \mathbf{Z}_N} q^{h^2} W_h.$$

Denote this sum by G. Similarly, we have the following equality in $\mathcal{S}(S^3; q)$:



$$= \Big(\sum_{k \in \mathbf{Z}_N} q^{(k+n)^2} W_k \Big)$$



for any $n > 0$. Thus, the condition (3) from Lickorish method (see Proposition 4.3) is equivalent to the following system of equations:

$$
\left\{
\begin{array}{ccccccccc}
q^0 W_0 & + & q^1 W_1 & + & q^4 W_2 & +\ldots+ & q^{(N-1)^2} W_{N-1} & = G \\
q^1 W_0 & + & q^4 W_1 & + & q^9 W_2 & +\ldots+ & q^0 W_{N-1} & = G \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots \\
q^{(N-1)^2} W_0 & + & q^0 W_1 & + & q^1 W_2 & +\ldots+ & q^{(N-2)^2} W_{N-1} & = G
\end{array}
\right. \tag{4.1}
$$

Notice, that

$$
Q = \begin{pmatrix} 1 & q & q^4 & \cdots & q^{(N-1)^2} \\ q & q^4 & q^9 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ q^{(N-1)^2} & 1 & q & \cdots & q^{(N-2)^2} \end{pmatrix}
$$

is a (-1)-circulant matrix (for circulant and g-circulant matrices see [Da]). Consider the following unitary matrix

$$
\Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \end{pmatrix}.
$$

The product $\Gamma \cdot Q$ is a circulant matrix with eigenvalues

$$
\lambda_c = 1 + q e^{\frac{2\pi i c}{N}} + q^4 e^{\frac{4\pi i c}{N}} + \ldots + q^{(N-1)^2} e^{\frac{2(N-1)\pi i c}{N}} = \sum_{j=0}^{N-1} (e^{\frac{2\pi i}{N}})^{bj^2+cj},
$$

$$
c = 0, \ldots, N-1.
$$

Here we assume that $q = e^{\frac{2\pi i b}{N}}$ for some b coprime to N. For details see [Da]. Thus, every λ_c is a generalized Gaussian sum. If N is odd then $\lambda_c \neq 0$ (see [LL]). Hence, $\det Q \neq 0$. It follows, that if N is odd then there is only one linearly independent solution of (4.1), namely $W_0 = W_1 = \ldots = W_{N-1} = W$. Notice, that if $W = 1$, then the resulting invariant $Inv_\Omega(M_L)$ coincides with the invariant defined in Theorem 4.4, and the result follows from Proposition 4.6, with $C = |G_N(q)|$. Therefore, if $W \neq 1$, then $Inv_\Omega(M_L)$ differs from the invariant in Theorem 4.4 by the factor $W^{\beta_1(M_L)}$ and we need to take $C = W \cdot |G_N(q)|$.

Now consider the case $N = 2 \pmod 4$. Let $N = 4h+2$. Notice, that $q^{(N/2)^2} = q^{hN+2h+1} = q^{N/2} = -1$. This implies that the first line of (4.1) is

$$
W_0 + qW_1 + \ldots + q^{(2h)^2} W_{2h} - W_{2h+1} - qW_{2h+2} - \ldots - q^{(2h)^2} W_{N-1} = G,
$$

and the $(2h+1)$-th line is

$$-W_0 - qW_1 - \ldots - q^{(2h)^2}W_{2h} + W_{2h+1} + qW_{2h+2} + \ldots + q^{(2h)^2}W_{N-1} = G.$$

Therefore, $G = 0$, which contradicts condition (2) of Lickorish method. This implies that the case $N = 2 \pmod 4$ is impossible.

The only case left is $N = 0 \pmod 4$. Let $N = 4h$. Then $q^{(2h+i)^2} = q^{i^2}$. Therefore, for any n-component framed link L we have

$$\frac{< \omega_{i+2th} >_L}{< \omega_{i+2th} >_{T_1}} = \frac{< \omega_i >_L}{< \omega_i >_{T_1}},$$

whenever $< \omega_i >_{T_1} \neq 0$ and $< \omega_{i+2th} >_{T_1} \neq 0$. Thus, with the notation

$$W'_i = \sum_{t \in \mathbf{Z}} < \omega_{i+2th} >_{T_1},$$

similarly as before, we have

$$< \Omega >_L = \sum_{l \in \mathbf{Z}^n_{2h}} q^{l^t A l} \cdot \prod_{s=1}^{n} W'_{l_s}.$$

Proceeding as earlier, we get the following system:

$$(4.2) \qquad \begin{cases} q^0 W'_0 & + & q^1 W'_1 & + & q^4 W'_2 & + \ldots + & q^{(2h-1)^2} W'_{2h-1} & = G \\ q^1 W'_0 & + & q^4 W'_1 & + & q^9 W'_2 & + \ldots + & q^0 W'_{2h-1} & = G \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ q^{(2h-1)^2} W'_0 & + & q^0 W'_1 & + & q^1 W'_2 & + \ldots + & q^{(2h-2)^2} W'_{2h-1} & = G \end{cases}$$

As before, consider the corresponding matrices Q and Γ. This time they will be of size $2h$. Again, $\Gamma \cdot G$ is a circulant matrix with the eigenvalues

$$\lambda'_c = 1 + q e^{\frac{2\pi i c}{2h}} + q^4 e^{\frac{4\pi i c}{2h}} + \ldots + q^{(2h-1)^2} e^{\frac{2(2h-1)\pi i c}{2h}} = \sum_{j=0}^{2h-1} (e^{\frac{2\pi i}{4h}})^{bj^2 + 2cj},$$

$$c = 0, \ldots, 2h-1.$$

Notice, that the number $2c$ is even. Therefore, by Theorem 2.2 from [LL], $\lambda'_c \neq 0$. Thus, in case $N = 0 \pmod 4$ there is only one linearly independent solution of (4.2), namely $W'_0 = W'_1 = \ldots = W'_{N-1} = W'$. If $W' = 1$ then the resulting invariant $Inv_\Omega(M_L)$ coincides with the invariant from the Theorem 4.4. Hence, our theorem follows from Proposition 4.6, with $C = W'|2G_{\frac{N}{2}}(q)|$. □

Remark 4.8. It follows from Theorems 4.4 and 4.6 and from Remark 4.5 that the previous theorem could be formulated in terms of the skein module $\mathcal{S}_2$. Moreover, the result of the theorem implies that if q is a root of unity then Lickorish method on $\mathcal{S}$ produces the same 3-manifold invariants as on $\mathcal{S}_2$.

REMARK 4.9. The proof of Theorem 4.7 implies that in order to show that Murakami-Ohtsuki-Okada invariants are the only 3-manifold invariants that can be obtained using Lickorish method on the skein module $\mathcal{S}$, it suffices to prove that the following infinite system

$$\left\{\begin{array}{ccccccc} q^{m^2}W_m & + & q^{(m+1)^2}W_{m+1} & +\ldots+ & q^{M^2}W_M & = G \\ q^{(m+1)^2}W_m & + & q^{(m+2)^2}W_{m+2} & +\ldots+ & q^{(M+1)^2}W_M & = G \\ \vdots & & \vdots & & \vdots & \vdots \end{array}\right. \tag{4.3}$$

has a solution in variables $W_m, \ldots, W_M$ only if q is a root of unity.

CONJECTURE. *We conjecture that indeed, the system (4.3) has a solution in variables $W_m, \ldots, W_M$ only if q is a root of unity. That is, no other invariant (except the MOO-invariants) can be obtained applying Lickorish method to the skein module $\mathcal{S}$.*

REMARK 4.10. It is proven in [Fe] that in order to apply Lickorish method to the skein module $\mathcal{S}_2$, q must be a root of unity.

4.4 Deloup invariants and dichromatic skein module

By *Deloup invariants* we mean the generalization of Murakami-Ohtsuki-Okada invariants studied in [De]. Essentially, these invariants were introduced by V. Turaev via a simple example of modular categories (see [T2] p. 29; see, also, [De] Appendix A).

Let M be a closed oriented 3-manifold and let X be a smooth compact connected oriented 4-manifold such that $\partial X = M$ and $H_1(X) = 0$ (such a manifold always exists by Rokhlin's theorem). Then $H_2(X)$ has no torsion. Denote by $B_X\colon H_2(X) \times H_2(X) \to \mathbf{Z}$ the intersection pairing on X. Let G be a finite abelian group. Fix a quadratic form $\xi\colon G \to \mathbf{Q}/\mathbf{Z}$, i.e., a function from G to $\mathbf{Q}/\mathbf{Z}$ satisfying $\xi(nx) = n^2\xi(x)$ for any $n \in \mathbf{Z}$ and $x \in G$ and such that the function defined by $b_\xi(a,b) = \xi(a+b) - \xi(a) - \xi(b)$ is a symmetric bilinear form on G, called the *bilinear form associated to* ξ. Let $\widehat{b}_\xi\colon G \to Hom(G, \mathbf{Q}/\mathbf{Z})$ denote the adjoint homomorphism of b_ξ. Set

$$\gamma(G,\xi) = |G|^{-\frac{1}{2}}|\ker\, \widehat{b}_\xi|^{-\frac{1}{2}} \sum_{x \in G} e^{2\pi i \xi(x)}.$$

For any (G,ξ) such that $\gamma(G,\xi) \neq 0$, *Deloup invariant* is defined as

$$\tau(M;G,\xi) = \overline{\gamma(G,\xi)}^{\sigma(B_X)} |G/\ker\, \widehat{b}_\xi|^{-\frac{\beta_2(X)}{2}} \sum_{x \in G \otimes H_2(X)} e^{2\pi i (\xi \otimes B_X)(x)}.$$

Here bar denotes complex conjugation, σ is the signature and $\beta_2(X)$ is the rank of $H_2(X)$. $\xi \otimes B_X$ denotes the $\mathbf{Q}/\mathbf{Z}$-valued quadratic form on $G \otimes H_2(X)$ defined by

$$(\xi \otimes B_X)(\sum_j x_j \otimes y_j) = \sum_j \xi(x_j) B_X(y_j, y_j) + \sum_{j<k} b_\xi(x_j, x_k) B_X(y_j, y_k),$$

where $x_j \in G, y_j \in H_2(X)$.

Remark 4.11.([De]) Let M be a 3-manifold obtained by the surgery on a framed link L with m components. Let A be the linking matrix of L. Then

$$\tau(M; G, \xi) = \overline{\gamma(G, \xi)}^{\sigma(A)} |G/\ker \widehat{b}_\xi|^{-\frac{m}{2}} \sum_{x \in G \otimes \mathbf{Z}^m} e^{2\pi i (\xi \otimes A)(x)},$$

where $\xi \otimes A \colon G \otimes \mathbf{Z}^m \to \mathbf{Q}/\mathbf{Z}$ denotes the tensor product of ξ and the symmetric bilinear form defined by A.

Remark 4.12.([De]) Fix $N > 1$ and let q be an N-th primitive root of unity if N is odd and $2N$-th primitive root of unity if N is even. Then

$$MOO_N(M; q) = \tau(M; \mathbf{Z}_N, \xi),$$

where the quadratic form $\xi \colon G \to \mathbf{Q}/\mathbf{Z}$ is chosen so that $q = e^{2\pi i \xi(1)}$. This proves that Deloup invariants generalize Murakami-Ohtsuki-Okada invariants.

Let u, v be two singular 1-cycles in M respectively representing torsion elements $[u], [v]$ in $H_1(M)$. There exists $n \in \mathbf{Z}$ such that $nv = \partial C$, where C is a singular 2-chain in M. Then the *linking pairing* $\mathcal{L}_M$ is the bilinear pairing defined on Tors $H_1(M)$ by

$$\mathcal{L}_M([u], [v]) = \frac{u \cdot C}{n} \pmod 1,$$

where $u \cdot C$ denotes the intersection number of u and C in M.

Theorem 4.13.([De]) *If the pair (G, ξ) is fixed, then τ is completely defined by the following data:*

1) the first Betti number;

2) the linking pairing $\mathcal{L}_M$ on Tors $H_1(M)$*, considered up to isomorphism.*

Remark 4.14. The fact that Murakami-Ohtsuki-Okada invariants are determined by the first Betti number and the linking pairing was noticed in [MOO] and easily follows from the property that if $H_1(M)$ is finite then a matrix representing $\mathcal{L}_M$ is equal to $-A^{-1}$, where A is the linking matrix of a framed link L such that M is obtained by the surgery on L. We will study this relation in Appendix C.

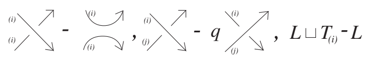
Recently, F. Deloup and C. Gille proved that, in fact, Deloup invariants carry exactly the same topological information about 3-manifolds as the first Betti number and $\mathcal{L}_M$ [DG]. Before we state their main result, we need to introduce two particular families of quadratic forms. Given two non-zero integers a and b, we shall denote by $\left(\frac{a}{b}\right) : \mathbf{Z}_b \to \mathbf{Q}/\mathbf{Z}$ the quadratic form sending 1 to $\frac{a}{b} \in \mathbf{Q}/\mathbf{Z}$. For any $m \in \mathbf{Z}_+$ we define a non-degenerate quadratic form $Q_m \colon \mathbf{Z}_{2^m} \times \mathbf{Z}_{2^m} \to \mathbf{Q}/\mathbf{Z}$ by $Q_m(a,b) = \frac{ab}{2^m} \in \mathbf{Q}/\mathbf{Z}$.

Theorem 4.15.([DG]) *Let M and M' be two closed connected oriented 3-manifolds. The following assertions are equivalent:*

1) *$\beta_1(M) = \beta_1(M')$ and $(\mathrm{Tors}\ H_1(M), \mathcal{L}_M) \simeq (\mathrm{Tors}\ H_1(M'), \mathcal{L}_{M'})$ (i.e. the linking pairings are isomorphic);*
2) *for all pairs (G,ξ), $\tau(M;G,\xi) = \tau(M';G,\xi)$;*
3) *for all prime p and for all $m \in \mathbf{Z}_+$, $\tau(M;\mathbf{Z}_{p^m},\left(\frac{1}{p^m}\right)) = \tau(M';\mathbf{Z}_{p^m},\left(\frac{1}{p^m}\right))$ and $\tau(M;\mathbf{Z}_{2^m} \times \mathbf{Z}_{2^m}, Q_m) = \tau(M';\mathbf{Z}_{2^m} \times \mathbf{Z}_{2^m}, Q_m)$.*

Notice that $\tau(M;\mathbf{Z}_{p^m},\left(\frac{1}{p^m}\right)) = MOO_{p^m}(M; e^{\frac{2\pi i}{p^m}})$ (see Remark 4.12). Thus, the only essential difference between Murakami-Ohtsuki-Okada invariants and Deloup invariants is contained in the family $\tau_{2,m}(M) = \tau(M;\mathbf{Z}_{2^m} \times \mathbf{Z}_{2^m}, Q_m)$. We already have shown how Murakami-Ohtsuki-Okada invariants can be obtained using Lickorish's construction on the skein module $\mathcal{S}_2$ (see Theorem 4.4, Remark 4.5, and Proposition 4.6). Is it possible to construct the invariants $\tau_{2,m}(M)$ using the same techniques? In order to do this we need to introduce another skein module.

Definition. We will say that a link L is *bicolored* if each component of L is marked by either 0 or 1. Let $\mathcal{L}^{col}$ denote the set of oriented bicolored links in an oriented 3-manifold M (including the empty link $\emptyset$). Let R be a commutative ring with unity. Let us consider the submodule S^{col} of $R\mathcal{L}^{col}$ which is generated by the following skein expressions:



Here a number in parenthesis next to a component denotes its color, $L \in \mathcal{L}^{col}$, $L \sqcup T_{(i)}$ denotes an element of $\mathcal{L}^{col}$ such that its component $T_{(i)}$ is marked by i and bounds a disk D embedded in M, with $D \cap L = \emptyset$; $i,j \in \{0,1\}, i \neq j$. We define the *dichromatic skein module* as $\mathcal{D}(M;R,q) = R\mathcal{L}^{col}/S^{col}$. By $\mathcal{D}(M;q)$ we will mean the skein module $\mathcal{D}(M;\mathbf{Z}[q^{\pm 1}],q)$

Remark 4.16. The skein module $\mathcal{D}(S^3;q)$ has a natural algebra structure with multiplication defined by the disjoint union of links.

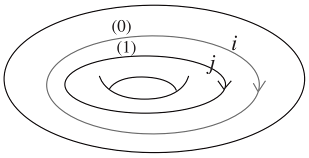
Proposition 4.17. $\mathcal{D}(S^3;q) \simeq \mathbf{Z}[q^{\pm 1}]$ *as algebras, with isomorphism defined by $\mu \colon T_n \mapsto 1$, where T_n is a trivial link of arbitrary colored n components, $n \geq 0$.*

Furthermore, for any link $L \in \mathcal{L}^{col}$, $\mu(L) = q^x$, where x is the sum of the linking numbers between the components of L having different colors.

PROOF. The proof is similar to the proof of Proposition 4.2. The inverse map to μ is defined by $\nu(q^x) = q^x \cdot \emptyset$, where $\emptyset$ denotes the empty link. $\square$

Fix an element $\Omega \in \mathcal{D}(D^2 \times S^1; q)$. As before, any framed oriented link $L \subset S^3$ with n components defines an oriented embedding of disjoint union of n solid tori $D^2 \times S^1$ into S^3, thus defines a multilinear map $\eta^L: \mathcal{D}(D^2 \times S^1; q)^n \to \mathcal{D}(S^3; q)$. By $< \Omega >_L$ we will denote the element of $\mathbf{Z}[q^{\pm 1}]$ equal to $\mu(\eta^L(\Omega \times \ldots \times \Omega))$. We can easily re-formulate Proposition 4.3 putting the skein module $\mathcal{D}$ in all the places where we mentioned the skein module $\mathcal{S}$. Also, in condition (3) instead of n parallel "plane" strings we need to consider n parallel strings n_0 of which are marked by 0 and n_1 marked by 1, $n = n_0 + n_1$, $n_0, n_1 > 0$. Again, if $\Omega \in \mathcal{D}(D^2 \times S^1)$ satisfies all three conditions (modified as above) from Proposition 4.3 (i.e., from Lickorish's method), then the resulting number $Inv_\Omega(M_L; q)$ is a 3-manifold invariant.

THEOREM 4.18. *Fix $t \geq 2$ and*

$\Omega = \sum_{i,j=0}^{t-1}$  $\in \mathcal{D}(D^2 \times S^1; q)$

Let q be a primitive t-th root of unity. Then the function $Inv_\Omega(M_L)$ is a non-trivial invariant of closed oriented 3-manifolds. Furthermore, if $q = e^{\frac{2\pi s i}{t}}$, with $(s,t) = 1$, then

$$Inv_\Omega(M_L) = t^{\beta_1(M_L)-k} < \Omega >_L = t^{\beta_1(M_L)} \tau(M_L; \mathbf{Z}_t \times \mathbf{Z}_t, Q),$$

where k is the number of components of L and $Q(a,b) = \frac{sab}{t} \in \mathbf{Q}/\mathbf{Z}$. In particular, if $q = e^{\frac{2\pi i}{2m}}$, then $Inv_\Omega(M_L) = t^{\beta_1(M_L)} \tau_{2,m}(M_L)$.

PROOF. Let A be the linking matrix of L and let $\theta(z,w)$ be a term in the sum $\eta_L(\Omega, \ldots, \Omega)$ obtained by taking $(z_i + w_i)$ parallel copies of the component l_i of L, where z_i copies are marked by 0 and w_i copies are marked by 1, $z = (z_1, \ldots, z_k) \in \mathbf{Z}_t^k, w = (w_1, \ldots, w_k) \in \mathbf{Z}_t^k$. Then $\mu(\theta(z,w)) = q^{z^t A w}$. Thus,

$$< \Omega >_L = \sum_{(z,w) \in \mathbf{Z}_t^k \times \mathbf{Z}_t^k} q^{z^t A w}.$$

Now we can check that our choice of q and Ω satisfies the three conditions of Lickorish's method. Condition (3) is equivalent to the equation

$$\sum_{i,j \in \mathbf{Z}_t} q^{(i+n_0)(j+n_1)} = \sum_{i,j \in \mathbf{Z}_t} q^{ij}, \quad n_0, n_1 \in \mathbf{N}, n_0 + n_1 \geq 1,$$

which is obviously true. Since q is a primitive t-th root of unity, we have

$$< \Omega >_{\alpha^+} = < \Omega >_{\alpha^-} = \sum_{i,j \in \mathbf{Z}_t} q^{ij} = \sum_{i \in \mathbf{Z}_t} \sum_{j \in \mathbf{Z}_t} (q^i)^j = t \neq 0,$$

thus, condition (2) of Lickorish's method also holds. Moreover, this implies that $< \Omega >_{\alpha^+}^{-b_+} < \Omega >_{\alpha^-}^{-b_-} = t^{\beta_1(M_L)-k}$.

In order to show condition (1), consider the framed link L' obtained from L by changing the orientation of the last component l_k. In the above sum for $< \Omega >_L$, the effect of changing the orientation for l_k is the same as after changing each vector $(z_1, \ldots, z_k, w_1, \ldots, w_k)$ to $(z_1, \ldots, z_{k-1}, t - z_k, w_1, \ldots, w_{k-1}, t - w_k)$, which does not change the total set of the vectors. Thus, $< \Omega >_L = < \Omega >_{L'}$, and condition (1) of Lickorish's method holds. This implies that the resulting number $Inv_\Omega(M_L)$ is indeed an invariant of closed oriented 3-manifolds. It is easy to check that this invariant is not trivial.

To prove the second part of the theorem we need to show that

$$\tau(M_L; \mathbf{Z}_t \times \mathbf{Z}_t, Q) = t^{-k} \sum_{(z,w) \in \mathbf{Z}_t^k \times \mathbf{Z}_t^k} q^{z^t A w}.$$

We will use the formula for τ from Remark 4.11. First of all, notice that $\gamma(G; \xi)$ is either 8th root of unity or 0 (see [De]). Therefore, since

$$\sum_{x \in \mathbf{Z}_t \times \mathbf{Z}_t} e^{2\pi i Q(x)} = \sum_{l,j \in \mathbf{Z}_t} q^{lj} = t \in \mathbf{R},$$

the number $\gamma(\mathbf{Z}_t \times \mathbf{Z}_t, Q)$ must be 1. Thus,

$$\tau(M_L; \mathbf{Z}_t \times \mathbf{Z}_t, Q) = t^{-k} \sum_{x \in (\mathbf{Z}_t \times \mathbf{Z}_t) \otimes \mathbf{Z}^k} e^{2\pi i (Q \otimes A)(x)}.$$

Every element of $(\mathbf{Z}_t \times \mathbf{Z}_t) \otimes \mathbf{Z}^k$ can be represented by an element of the set $S = \{((1,0), z) + ((0,1), w) \quad | \quad z, w \in \mathbf{Z}_t^k\}$, moreover, any two elements of S represent different elements of the tensor product. Furthermore, $(Q \otimes A)(((1,0), z) + ((0,1), w)) = Q(1,0)A(z,z) + Q(0,1)A(w,w) + Q(1,1)A(z,w) + Q(0,0)A(w,z) = Q(1,1)A(z,w) = \frac{s}{t} \cdot z^t A w$. The result follows. $\square$

4.5 Structure of the dichromatic skein module

In this section we will study structure of the dichromatic skein module and show its relation with the q-analog of the first homology skein module $\mathcal{S}_2$.

First, we need to introduce some notations. Let M be an oriented 3-manifold. Every oriented link in M represents an element of $H_1(M)$, and every element of

$H_1(M)$ can be represented by an oriented link in M. For every $\alpha \in H_1(M)$ fix an oriented link L_α representing it. Denote the set of such links by $\mathbf{L}$, that is $\mathbf{L} = \{L_\alpha \subset M : \alpha \in H_1(M)\}$. For every link $L_\alpha \in \mathbf{L}$ fix an arbitrary framing and denote the resulting framed link by L^f_α. Set $\mathbf{L}^f = \{L^f_\alpha : L_\alpha \in \mathbf{L}\}$. Notice that $\mathbf{L}^f$ naturally embeds into $R\mathcal{L}^f_*$, therefore, with an abuse of notations, we will treat $\mathbf{L}^f$ as a subset of $\mathcal{S}_2(M; R, q)$. Take a pair of links $L_\alpha, L_\beta \in \mathbf{L}$. Their disjoint union, as a link in M, is not well defined, and may be represented by infinitely many different links. Fix one of them and denote it by $L_\alpha \& L_\beta$. Set $\mathbf{L}^2 = \{L_\alpha \& L_\beta : L_\alpha, L_\beta \in \mathbf{L}\}$. Sometimes, with an abuse of notations, we will consider $\mathbf{L}^2$ as a set of bi-colored links, meaning that for a link $L_\alpha \& L_\beta \in \mathbf{L}^2$ the components of L_α are marked by 0 and the components of L_β are marked by 1. In this case, there is a natural embedding $\mathbf{L}^2 \hookrightarrow R\mathcal{L}^{col}$, and with an abuse of notations, we will treat $\mathbf{L}^2$ as a subset of $\mathcal{D}(M; R, t)$. For an oriented (framed) link $L \subset M$, by $\widehat{L}$ we will denote the element of $\mathbf{L}$ ($\mathbf{L}^f$) representing the same element of $H_1(M)$ as L. Given a bi-colored link $L \subset M$, by $L_{(0)}$ we will denote the sublink of L colored by 0, and by $L_{(1)}$ we will denote the sublink of L colored by 1. Notice that in our notations, $\widehat{L_{(0)}} \& \widehat{L_{(1)}} \in \mathbf{L}^2$.

Now, let us recall some definitions and the main theorem from [Pr4].

Definition. ([Pr4]) Let $\phi\colon H_1(M) \times H_2(M) \to \mathbf{Z}$ denote the bilinear form of the intersection of 1-cycles with 2-cycles in M. We define $T(H_1(M))$ to be the left kernel of ϕ, that is $T(H_1(M)) = \{\alpha \in H_1(M) : \phi_\alpha(\beta) = \phi(\alpha, \beta) = 0 \text{ for any } \beta \in H_2(M)\}$. In particular, for a closed 3-manifold M, $T(H_1(M))$ is the torsion part of $H_1(M)$. We define *multiplicity* of $\alpha \in H_1(M)$, $m(\alpha)$, to be 0 if $\alpha \in T(H_1(M))$ and otherwise $m(\alpha)$ is defined as the positive generator of $im\ \phi_\alpha(H_2(M))$. [1]

Theorem 4.19 ([Pr4]) *Let the R-homomorphism $g\colon RH_1(M) \to \mathcal{S}_2(M; R, q)$ be given by $g(\alpha) = L^f_\alpha \in \mathbf{L}^f \hookrightarrow \mathcal{S}_2(M; R, q)$. Then g descends to the isomorphism*

$$\widehat{g}\colon RH_1(M)/I \to \mathcal{S}_2(M; R, q),$$

where I is the ideal generated by the elements $(q^{2m(\alpha)} - 1)\alpha$, $\alpha \in H_1(M)$.

Later in this section we will prove a similar result for the dichromatic skein module. Let $m(\alpha, \beta)$ denote the greatest common divisor of $m(\alpha)$ and $m(\beta)$.

Lemma 4.20. *Let L be an oriented bi-colored link in M. Then $(t^{m(\alpha,\beta)} - 1)L = 0$ in $\mathcal{D}(M; R, t)$, where $\alpha, \beta \in H_1(M)$ are such that $\widehat{L_{(0)}} = L_\alpha \in \mathbf{L}$ and $\widehat{L_{(1)}} = L_\beta \in \mathbf{L}$.*

Proof. The main idea of our proof is the same as in the proof of Lemma 2.4 of [Pr4]. The definition of $m(\cdot)$ implies that there are two closed oriented surfaces F_0 and F_1 embedded in M such that $alg(L_{(0)}, F_0) = m(\alpha)$ and $alg(L_{(1)}, F_1) = m(\beta)$ (see, for instance, [Pr4]). Let T be a marked by 1 trivial knot on F_0 that bounds a

[1]The fact that $H_2(M)$ is a free abelian group implies that $im\ \phi_\alpha(H_2(M))$ is an infinite cyclic group.

disk $D \subset F_0$ disjoint from $L_{(0)}$. Let $L' = L_{(0)} \cup T$. Obviously, $L' = L_{(0)} = L_\alpha$ in $\mathcal{D}(M; R, t)$. On the other hand, $T = T'$ in $\mathcal{D}(M - L_{(0)}; R, t)$, where T' is a trivial knot on F_0 marked by 1, that bounds a disk $D' \subset F_0$ such that $int(D')$ contains all the points of intersection of F_0 and $L_{(0)}$ (see Lemma 1.3 of [Pr4]). Now we can take T' "out" of $L_{(0)}$ by crossing changes, thus $L_{(0)} \cup T' = t^{m(\alpha)} L_{(0)} \cup T$. This implies that $L = t^{m(\alpha)} L$ in $\mathcal{D}(M; R, t)$. Similarly, $L = t^{m(\beta)} L$ in $\mathcal{D}(M; R, t)$, which implies the result. □

Lemma 4.21. *For all $\alpha, \beta \in H_1(M)$, $m(\alpha \cdot \beta) = m(\alpha, \beta)$.*

Proof. For every $C \in H_2(M)$ there are $x_1, x_2 \in \mathbf{Z}$ such that $\phi_{\alpha\beta}(C) = x_1 m(\alpha) + x_2 m(\beta)$. Therefore, from the definition of $m(\alpha, \beta)$ we have $\phi_{\alpha\beta}(C) = (x_1 \cdot \frac{m(\alpha)}{m(\alpha,\beta)} + x_2 \cdot \frac{m(\beta)}{m(\alpha,\beta)}) m(\alpha, \beta)$. □

Now we will introduce some more notations. Consider a subset of $R\mathcal{L}^f_*$ consisting of oriented framed links in M multiplied by elements of R, and denote it by $R \times \mathcal{L}^f_*$. It is easy to see that if two elements of $R \times \mathcal{L}^f_*$ are related by a finite sequence of the local moves $s_1^{\pm 1}$ from Fig. 4.1, then they represent the same element in $\mathcal{S}_2(M; R, q)$. We will also need the move s_2 from Fig. 4.1, which can be expressed as s_1^2.

Fig. 4.1.

Similarly, let $R \times \mathcal{L}^{col}$ be the subset of $R\mathcal{L}^{col}$ consisting of the oriented bi-colored links in M multiplied by elements of R. It is easy to see that two elements of $R \times \mathcal{L}^{col}$ represent the same element of $\mathcal{D}(M; R, t)$ iff they are related by a finite sequence of moves $d_1^{\pm 1}, d_2^{\pm 1}$, and $d_3^{\pm 1}$ from Fig. 4.2.

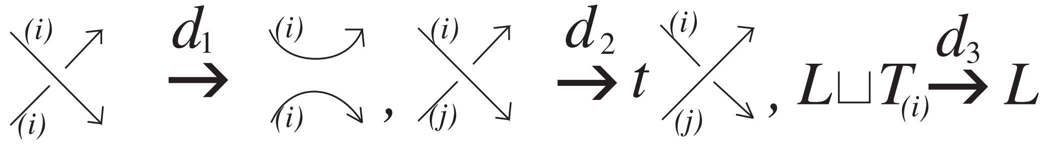


Fig. 4.2.

In the following definition we generalize the notion of linking number for links in S^3 to arbitrary oriented 3-manifolds.

Definition. Let L be an oriented bi-colored link and let $\alpha, \beta \in H_1(M)$ are such that $\widehat{L_{(0)}} = L_\alpha \in \mathbf{L}$ and $\widehat{L_{(1)}} = L_\beta \in \mathbf{L}$. Let $\widehat{L} = L_\alpha \&' L_\beta$ be an oriented bi-colored link such that $(\widehat{L})_{(0)} = L_\alpha$ and $(\widehat{L})_{(1)} = L_\beta$ and which is obtained from L by first using the move(s) d_3^{-1} (if needed, to add trivial component(s) $T_{(0)}$ and/or $T_{(1)}$ if $L_{(0)}$ and/or $L_{(1)}$ are empty), and then applying a finite number of moves $d_1^{\pm 1}$. After that, applying a finite number of moves $d_2^{\pm 1}$ to $\widehat{L}$, we transform it into $t^x(L_\alpha \& L_\beta)$ (in particular, $L = \widehat{L} = t^x(L_\alpha \& L_\beta)$ in $\mathcal{D}(M; R, t)$). We will denote such x by $LK(L)$.

THEOREM 4.22. *For every oriented bi-colored link $L \subset M$, the number $LK(L)$ is well defined modulo $m(\alpha, \beta)$, where $\alpha, \beta \in H_1(M)$ are such that $\widehat{L_{(0)}} = L_\alpha \in \mathbf{L}$ and $\widehat{L_{(1)}} = L_\beta \in \mathbf{L}$.*

PROOF. We will assume that neither $L_{(0)}$ nor $L_{(1)}$ are empty (otherwise introduce $T_{(0)}$ and/or $T_{(1)}$ using the move(s) d_3^{-1}, this operation is always well defined). In this case, there exists a finite sequence l_1 of moves $d_1^{\pm 1}$ transforming L into $\widehat{L} = L_\alpha \&' L_\beta$. Let p_1 be a finite sequence of moves $d_2^{\pm 1}$ that transforms $\widehat{L}$ into $t^{x_1}(L_\alpha \& L_\beta)$. Consider another sequence l_2 of moves $d_1^{\pm 1}$, that transforms L into $\widetilde{L} = L_\alpha \&'' L_\beta$ ($(\widetilde{L})_{(0)} = L_\alpha$ and $(\widetilde{L})_{(1)} = L_\beta$), and let p_2 be a finite sequence of moves $d_2^{\pm 1}$ that transforms $\widetilde{L}$ into $t^{x_2}(L_\alpha \& L_\beta)$. Consider the oriented framed links $L_\alpha^f \&' L_\beta^f$, $L_\alpha^f \&'' L_\beta^f$ and $L_\alpha^f \& L_\beta^f$ obtained by replacing the sublinks L_α and L_β in $L_\alpha \&' L_\beta$, $L_\alpha \&'' L_\beta$ and $L_\alpha \& L_\beta$ by the corresponding framed links $L_\alpha^f, L_\beta^f \in \mathbf{L}^f$. Let L^f be an oriented framed link whose core is L and such that $L^f = L_\alpha^f \&' L_\beta^f$ in $\mathcal{S}_2(M; R, q)$. By $L_{(0)}^f$ and $L_{(1)}^f$ denote the sublinks of L^f whose cores are $L_{(0)}$ and $L_{(1)}$, respectively. Let l_i', p_i' be the sequences of moves $s_1^{\pm 1}$, $s_2^{\pm 1}$ obtained from the sequences l_i, p_i respectively, i=1,2, by replacing the moves $d_1^{\pm 1}$ and $d_2^{\pm 1}$ with the corresponding moves $s_1^{\pm 1}$ and $s_2^{\pm 1}$. This implies that

$$L^f \xrightarrow{l_1'} L_\alpha^f \&' L_\beta^f \xrightarrow{p_1'} q^{2x_1}(L_\alpha^f \& L_\beta^f), \text{ and}$$

$$L^f \xrightarrow{l_2'} q^y(L_\alpha^f \&'' L_\beta^f) \xrightarrow{p_2'} q^{2x_2+y}(L_\alpha^f \& L_\beta^f),$$

for some $y \in \mathbf{Z}$. Therefore,

$$q^{2x_1}(L_\alpha^f \& L_\beta^f) = L^f = q^{2x_2+y}(L_\alpha^f \& L_\beta^f)$$

in $\mathcal{S}_2(M; R, q)$, and by Theorem 4.19 we have

$$2x_1 = 2x_2 + y \pmod{2m(\alpha \cdot \beta)}.$$

Each of the sequences l_i', $i = 1, 2$, can be split into two subsequences l_{i0}' and l_{i1}', where l_{i0}' are the moves applied to $L_{(0)}^f$ and l_{i1}' are the moves applied to $L_{(1)}^f$. This implies that there are $y_1, y_2 \in \mathbf{Z}$ such that $y = y_1 + y_2$ and $L_{(0)}^f \xrightarrow{l_{20}'} q^{y_1} L_\alpha^f$, $L_{(1)}^f \xrightarrow{l_{21}'} q^{y_2} L_\beta^f$. Therefore, $L_\alpha^f = q^{y_1} L_\alpha^f$ and $L_\beta^f = q^{y_2} L_\beta^f$ in $\mathcal{S}_2(M; R, q)$. By Theorem 4.19, we have $y_1 = 0 \pmod{2m(\alpha)}$ and $y_2 = 0 \pmod{2m(\beta)}$. Thus,

$$y = y_1 + y_2 = 0 \pmod{2m(\alpha, \beta)}.$$

By Lemma 4.21 we have $m(\alpha, \beta) = m(\alpha \cdot \beta)$, therefore $x_1 = x_2 \pmod{m(\alpha, \beta)}$. $\square$

REMARK 4.23. In a private conversation with the author, Uwe Kaiser suggested the following alternative definition of the number $LK(L)$. Consider two oriented surfaces $F_0, F_1 \hookrightarrow M\times[0,1]$ such that $\partial F_0 = L_{(0)} \cup L_\alpha$, $\partial F_1 = L_{(1)} \cup L_\beta$, $(F_0 \cup F_1)\cap M\times\{0\} = L$, and $(F_0\cup F_1)\cap M\times\{1\} = L_\alpha \& L_\beta \in \mathbf{L}^2$. Here the orientation of L coincides with the orientation of $F_0 \cup F_1$, and the orientation of $L_\alpha \& L_\beta$ is the opposite one. Then $LK(L)$ can be defined as the algebraic intersection number between F_0 and F_1. The fact that $LK(L)$ is well defined modulo $m(\alpha,\beta)$ can be proven by the arguments analogous to the arguments in [Ka], with one of the most important technical tools being Lemma 2.4.3.

The following theorem is an analogue of Theorem 4.19 for the dichromatic skein module.

THEOREM 4.24. *Let the R-homomorphism $f\colon R(H_1(M)\times H_1(M)) \to \mathcal{D}(M;R,t)$ be given by $f(\alpha,\beta) = L_\alpha \& L_\beta \in \mathbf{L}^2 \hookrightarrow \mathcal{D}(M;R,t)$, for all $\alpha,\beta\in H_1(M)$. Then f descends to the isomorphism*

$$\widehat{f}\colon R(H_1(M)\times H_1(M))/I_2 \to \mathcal{D}(M;R,t),$$

where I_2 is the ideal generated by the elements $(t^{m(\alpha,\beta)}-1)(\alpha,\beta)$, $\quad \alpha,\beta\in H_1(M)$.

PROOF. Every element of $\mathcal{D}(M;R,t)$ can be presented as a linear combination of elements of $\mathbf{L}^2$ with coefficients in R. This and Lemma 4.20 imply that $\widehat{f}$ is a well defined epimorphism. It remains to prove that $\widehat{f}$ is a monomorphism. To show this, we will construct the inverse map $\mathcal{D}(M;R,t)\to R(H_1(M)\times H_1(M))/I_2$. First, define an R-homomorphism $h\colon R\mathcal{L}^{col} \to R(H_1(M)\times H_1(M))/I_2$ by $h(L) = t^{LK(L)}(\alpha,\beta)$, where $L\in\mathcal{L}^{col}$, and $\alpha,\beta\in H_1(M)$ are such that $\widehat{L_{(0)}} = L_\alpha \in \mathbf{L}$ and $\widehat{L_{(1)}} = L_\beta\in\mathbf{L}$. By Theorem 4.22, $LK(L)$ is well defined modulo $m(\alpha,\beta)$, thus h is well defined. We will show now that h descends to an R-homomorphism $\widehat{h}\colon\mathcal{D}(M;R,t)\to R(H_1(M)\times H_1(M))/I_2$. Indeed, if an oriented bi-colored link $L'\subset M$ can be obtained from L by one move d_1, then $\widehat{L} = \widehat{L'}$, thus $h(L) = h(L')$. Let $tL''\in R\times\mathcal{L}^{col}$ be obtained from L by applying one local move d_2. In this case $\widehat{L''_{(0)}} = L_\alpha$, $\widehat{L''_{(1)}} = L_\beta$, and $LK(L'') = LK(L)-1 \pmod{m(\alpha,\beta)}$. Thus, $h(tL'') = t\cdot t^{LK(L)-1}(\alpha,\beta) = h(L)$. Finally, notice that $LK(L\sqcup T_{(i)}) = LK(L)$, and moves $d_3^{\pm1}$ do not change the homological type of links, therefore $h(L\sqcup T_{(i)}) = h(L)$, $\quad i=1,2$.

It remains to check that $\widehat{h}$ is the inverse of $\widehat{f}$. Indeed,

$$\widehat{f}\circ\widehat{h}(L) = \widehat{f}(t^{LK(L)}(\alpha,\beta)) = t^{LK(L)}(L_\alpha\&L_\beta) = L$$

in $\mathcal{D}(M;R,t)$, and

$$\widehat{h}\circ\widehat{f}(\alpha,\beta) = \widehat{h}(L_\alpha\&L_\beta) = (\alpha,\beta)\in R(H_1(M)\times H_1(M))/I_2.$$

□

THEOREM 4.25. *The skein module $\mathcal{D}(M;R,q^2)$ is isomorphic to*

$\mathcal{S}_2(M;R,q) \otimes_R \mathcal{S}_2(M;R,q)$.

PROOF. Construct the balanced map

$$g\colon \mathcal{S}_2(M;R,q) \times \mathcal{S}_2(M;R,q) \to \mathcal{D}(M;R,q^2)$$

given by

$$g(r_1 L^f_\alpha, r_2 L^f_\beta) = r_1 r_2 (L_\alpha \& L_\beta),$$

for all $L^f_\alpha, L^f_\beta \in \mathbf{L}^f$, $r_1, r_2 \in R$, where $L_\alpha \& L_\beta \in \mathbf{L}^2$. It is well defined, because $q^{x_1} L^f_\alpha$ and $q^{x_2} L^f_\alpha$ represent the same framed link in $\mathcal{S}_2(M;R,q)$ iff $x_1 = x_2 \pmod{2m(\alpha)}$ (see Theorem 4.19), and

$$g((q^{2m(\alpha)} - 1) L^f_\alpha, L^f_\beta) = (q^{2m(\alpha)} - 1)(L_\alpha \& L_\beta) = L_\alpha \& L_\beta$$

in $\mathcal{D}(M;R,q^2)$, by Lemma 4.20. Therefore, g induces an epimorphism

$$\widehat{g}\colon \mathcal{S}_2(M;R,q) \otimes_R \mathcal{S}_2(M;R,q) \to \mathcal{D}(M;R,q^2).$$

(For definitions of balanced maps and tensor products see [BJN].) The inverse homomorphism

$$\widehat{g}^{-1}\colon \mathcal{D}(M;R,q^2) \to \mathcal{S}_2(M;R,q) \otimes_R \mathcal{S}_2(M;R,q)$$

can be constructed as follows:

$$\widehat{g}^{-1}(rL) = (r q^{2LK(L)} \widehat{L_{(0)}}^f) \otimes \widehat{L_{(1)}}^f$$

for all $rL \in R \times \mathcal{L}^{col}$. Since $LK(L)$ is well defined modulo $m(\alpha,\beta)$, where $L_\alpha = \widehat{L_{(0)}}$ and $L_\beta = \widehat{L_{(1)}}$, we need to check that

$$(q^{2m(\alpha,\beta)} L^f_\alpha) \otimes L^f_\beta = L^f_\alpha \otimes L^f_\beta.$$

Indeed, take $u, v \in \mathbf{Z}$ such that $u\, m(\alpha) + v\, m(\beta) = m(\alpha,\beta)$. Then we have

$$(q^{2m(\alpha,\beta)} L^f_\alpha) \otimes L^f_\beta = (q^{2m(\alpha)u} L^f_\alpha) \otimes (q^{2m(\beta)v} L^f_\beta) = L^f_\alpha \otimes L^f_\beta,$$

by Theorem 4.19.

It remains to confirm that $\widehat{g}^{-1}$ is indeed the inverse of $\widehat{g}$. Indeed,

$$\widehat{g}^{-1} \circ \widehat{g}((r_1 L^f_\alpha) \otimes (r_2 L^f_\beta)) = \widehat{g}^{-1}(r_1 r_2 (L_\alpha \& L_\beta)) = (r_1 r_2 L^f_\alpha) \otimes L^f_\beta = r_1 L^f_\alpha \otimes r_2 L^f_\beta$$

in $\mathcal{S}_2(M;R,q) \otimes_R \mathcal{S}_2(M;R,q)$; and

$$\widehat{g} \circ \widehat{g}^{-1}(rL) = \widehat{g}((r q^{2LK(L)} \widehat{L_{(0)}}^f) \otimes \widehat{L_{(1)}}^f) = r q^{2LK(L)} (\widehat{L_{(0)}} \& \widehat{L_{(1)}}) = rL$$

in $\mathcal{D}(M;R,q^2)$. $\square$

APPENDIX A
DIJKGRAAF-WITTEN INVARIANTS AND SINGULAR TRIANGULATIONS OF 3-MANIFOLDS

In this section we will show how to define Dijkgraaf-Witten invariants of 3-manifolds using singular triangulations and simple 2-polyhedra. For a definition of Dijkgraaf-Witten invariants of 3-manifolds using the regular triangulations see Section 3 of Chapter 3.

DEFINITION ([TV]). Let M be a 3-manifold. A family of continuous maps $\xi_\alpha\colon t^{d_\alpha} \to M$, of standard simplexes t^{d_α}, $d_\alpha \in \mathbf{Z}_+$, is called a *singular triangulation of* M, if

(1) all $\xi_\alpha|_{\mathrm{Int}\ t^{d_\alpha}}$ are embeddings;

(2) $\xi_\alpha(\mathrm{Int}\ t^{d_\alpha})$ are open cells of some CW-decomposition of X;

(3) for any face f of t^{d_α} the restriction $\xi_\alpha|_f$ can be obtained from some ξ_β by composition with a linear isomorphism $f \to t^{d_\beta}$.

Note that replacing $\xi_\alpha|_{\mathrm{Int}\ t^{d_\alpha}}$ by ξ_α in (1) and incorporating the condition that the intersection of any two simplexes is their common face convert this definition into a definition of triangulation. Notice that singular triangulations are much more convenient for practical reasons than the regular triangulations (see [TV]).

A *simple 2-skeleton of* M is a simple spine of $M - (kB^3)$, $k \geq 0$ (see 6.1.B from [TV]; for a definition of simple spine see, also, Section 1 of Chapter 1). A *special 2-skeleton of* M is a simple 2-skeleton of M which is also a special polyhedron, i.e., every 2-component is homeomorphic to a disk. Recall that a special spine of M is a simple spine of M which is also a special polyhedron (see Section 1 of Chapter 1).

Notice that the notions of singular triangulation and special 2-skeleton are dual to each other. Indeed, let T be a singular triangulation of M. Let $\widetilde{X}$ be the union of the closed barycentric stars of the edges of T. Let X be obtained from $\widetilde{X}$ by removing all the cells whose closure intersects with ∂M. It is easy to see that X is a special 2-skeleton of M. An inverse construction produces a singular triangulation from any special 2-skeleton of M. We will say that such T and X are dual to each other. Special polyhedra and special spines are well studied (see, for instance [M1], [M2], [Ca], [Fo], [TV]) and there are well structured tables of special spines of closed orientable 3-manifolds (see [M3], [Ov]).

The authors of [Wak] and [FG] also study Dijkgraaf-Witten invariants defined on special 2-skeletons, but their approach is valid only for the special 2-skeletons which are duals of regular triangulations.

Let T be a singular triangulation of a closed[1] oriented 3-manifold M and let n be the number of vertices of T. Orient the edges of T and fix a finite group G. A coloring

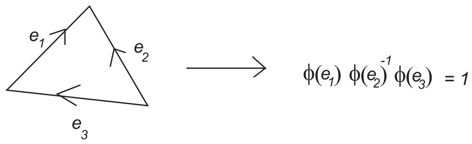
$$\phi\colon \{\text{edges of } T\} \to G$$

is called *admissible* ($\phi \in Adm_G T$) if for any face f of T with edges e_1, e_2, e_3 we have

$$\phi(e_1)^{\epsilon_1}\phi(e_2)^{\epsilon_2}\phi(e_3)^{\epsilon_3} = 1$$

in G. Here $\epsilon_i = \pm 1$ according to whether the orientation of the edge e_i is compatible with the orientation of f induced by the order (e_1, e_2, e_3) or not. Notice that this property (of the above product being equal to 1) is well defined, that is, it does not depend on the choice of the enumeration of the three edges of f.

EXAMPLE.



Now fix $\alpha \in H^3(G; \mathbf{Q}/\mathbf{Z})$. let t^3 be a 3-simplex of T. It is an image of a tetrahedron $\widetilde{t}$. Coloring ϕ of t^3 induces a coloring $\widetilde{\phi}$ of $\widetilde{t}$, and the orientation of the edges of t^3 induces an orientation of the edges of $\widetilde{t}$. Enumerate the vertices of $\widetilde{t}$ by 0, 1, 2, 3, in an arbitrary way (this is the main reason we need to consider $\widetilde{t}$, because t^3 could have less then 4 distinct vertices). Denote the orientation of $\widetilde{t}$ defined by the order of the vertices by *or1*. Let *or2* be the orientation of $\widetilde{t}$ coming from the orientation of the manifold M. Let e_{ij} be the edge connecting the vertices i and j of $\widetilde{t}$, where $i, j \in \{0, 1, 2, 3\}$. Let $\phi_{ij} = \phi(e_{ij})$ if the orientation of e_{ij} is compatible with the order of the vertices and $\phi_{ij} = \phi(e_{ij})^{-1}$, otherwise. Set

$$\widetilde{W}(t^3, \phi) = \alpha(\phi_{01}, \phi_{12}, \phi_{23})^{\epsilon(\widetilde{t})} \in \mathbf{Q}/\mathbf{Z},$$

where

$$\epsilon(\widetilde{t}) = \begin{cases} 1, & \text{if orientations } \mathit{or1} \text{ and } \mathit{or2} \text{ coincide;} \\ -1, & \text{otherwise.} \end{cases}$$

LEMMA A.1. *The number $\widetilde{W}(t^3, \phi)$ does not depend on the order of the vertices of $\widetilde{t}$.*
PROOF. It is enough to check only the following three permutations of the vertices of $\widetilde{t}$: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}$. Each of the permutations changes the orientation *or1* to the opposite one. Thus, it is enough to check that

[1]For simplicity, we will only define Dijkgraaf-Witten invariants on singular triangulations of closed 3-manifolds. It is possible to define relative Dijkgraaf-Witten invariants of 3-manifolds with boundary in a similar way as relative Turaev-Viro invariants were defined in [TV].

$\alpha(a^{-1}, ab, c) = \alpha(ab, b^{-1}, bc) = \alpha(a, bc, c^{-1}) = -\alpha(a, b, c)$, which follows from the fact that $\alpha \in H^3(G; \mathbf{Q}/\mathbf{Z})$. See, also, Lemma 3.1 from [Wak]. $\square$

We define Dijkgraaf-Witten invariant of M by the following formula

$$\widetilde{DW}_{G,\alpha}(M) = \frac{1}{|G|^{n-1}} \sum_{\phi \in Adm_G T} \prod_{t \in T} \widetilde{W}(t, \phi)$$

LEMMA A.2. *$\widetilde{DW}_{G,\alpha}(M)$ does not depend on the choice of orientation of the edges of T.*

PROOF. Let T and T' be two singular triangulations of M that are only different by orientation of an edge e. The sets $Adm_G T$ and $Adm_G T'$ are in one-to-one correspondence, given by the map $\phi \to \phi'$, where $\phi' = \phi|_{T-e}$ and $\phi'(e) = \phi(e)^{-1}$. Moreover, for any $t \in T$, $\widetilde{W}(t, \phi) = \widetilde{W}(t', \phi')$, where t' is the natural image of t in T'. $\square$

LEMMA A.3. *If T is a regular (non-singular) triangulation of a closed oriented 3-manifold M, then the number $\widetilde{DW}_{G,\alpha}(M)$ coincides with the number $DW_{G,\alpha}(M)$ defined in Chapter 3.*

PROOF. Order the vertices of T. Since $\widetilde{DW}_{G,\alpha}(M)$ does not depend on the way we orient the edges, orient them with respect to the enumeration of the vertices. The ordering of the vertices of T in the natural way induces an ordering of the vertices of each tetrahedron $\widetilde{t}$. $\square$

Before we prove that $\widetilde{DW}_{G,\alpha}(M)$ does not depend on a choice of a singular triangulation of M, we will show how to define Dijkgraaf-Witten invariants using a simple 2-skeleton of M.

Let X be a simple 2-skeleton of an oriented 3-manifold M (possibly, with boundary). This means that X is a simple spine of $M - (kB^3)$, $k \geq 0$. To avoid possible ambiguity in case when M is a closed manifold, we will always assume that k is the greatest integer such that X is a simple spine of $M - (kB^3)$. Orient the 2-components of X. By a *coloring of X* we mean a map

$$\varphi\colon \{\text{2-components of } X\} \to G$$

Take an edge e of X. Let c_1, c_2, c_3 be the 2-components of X adjusted to e. Since X is embedded into M, the order of the components induces an orientation on e. Set $\varepsilon_i = 1$ if the orientation of c_i is compatible with the orientation of e and $\varepsilon_i = -1$ otherwise, $i = 1, 2, 3$. We will say that *φ is admissible* ($\varphi \in Adm_G X$) if for any edge e we have

$$\varphi(c_1)^{\varepsilon_1} \varphi(c_2)^{\varepsilon_2} \varphi(c_3)^{\varepsilon_3} = 1$$

in G. The weight $W^*(v, \varphi)$ of a colored vertex v of X is defined as the weight $\widetilde{W}(t_v, \phi)$ of the dual tetrahedron t_v (a tetrahedron embedded into M whose faces are dual to the edges of X adjacent to v) with the coloring ϕ induced by the coloring φ, and with the orientation of the edges of t_v induced by the orientation of the corresponding 2-components of X. Define

$$DW^*_{G,\alpha}(M) = \frac{1}{|G|^{k-1}} \sum_{\phi \in Adm_G X} \prod_{i=1}^{m} W^*(v_i, \varphi),$$

where m is the number of the vertices of X.

THEOREM A.4.

1) *The number $DW^*_{G,\alpha}(M)$ does not depend on the choice of a simple 2-skeleton X and orientation of the 2-components of X, thus, it is an invariant of oriented 3-manifolds;*

2) *If $\partial M = \emptyset$, then $DW^*_{G,\alpha}(M)$ coincides with $\widetilde{DW}_{G,\alpha}(M)$, thus, $\widetilde{DW}_{G,\alpha}(M)$ does not depend on the choice of a singular triangulation and coincides with $DW_{G,\alpha}(M)$;*

3) *If $\partial M = \emptyset$, then $DW^*_{G,\alpha}(M) = DW_{G,\alpha}(M)$.*

PROOF.

1) Let X be a simple 2-skeleton of M. Then X is a simple spine of $M - (kB^3)$. Orient the 2-components of X. The same reasoning as in the proof of Lemma A.2 shows that $DW^*_{G,\alpha}(M)$ does not depend on the choice of the orientation. S. Matveev proved in [M1] that any two special spines of a 3-manifold can be transformed one into the other by a sequence of local moves $\mathcal{M}^{\pm 1}$ and $\mathcal{L}^{\pm 1}$ (for definitions of the local moves on simple polyhedra see Section 1 of Chapter 1). This easily implies that any two simple 2-skeletons of M can be transformed into each other by a sequence of the local moves $\mathcal{M}^{\pm 1}$, $\mathcal{L}^{\pm 1}$, and $\mathcal{B}^{\pm 1}$ (see Section 1 of Chapter 1, and [TV]). Thus, we only need to prove that $DW^*_{G,\alpha}(M)$ does not change if we apply one of the moves $\mathcal{M}^{\pm 1}$, $\mathcal{L}^{\pm 1}$, or $\mathcal{B}^{\pm 1}$ to X. Let X' be a simple polyhedron obtained from X by a move $\mathcal{L}$ (see Fig A.1). Fix $\varphi \in Adm_G X$. We may assume that the 2-components of X and X' affected by the transformation are oriented and colored as in the picture.

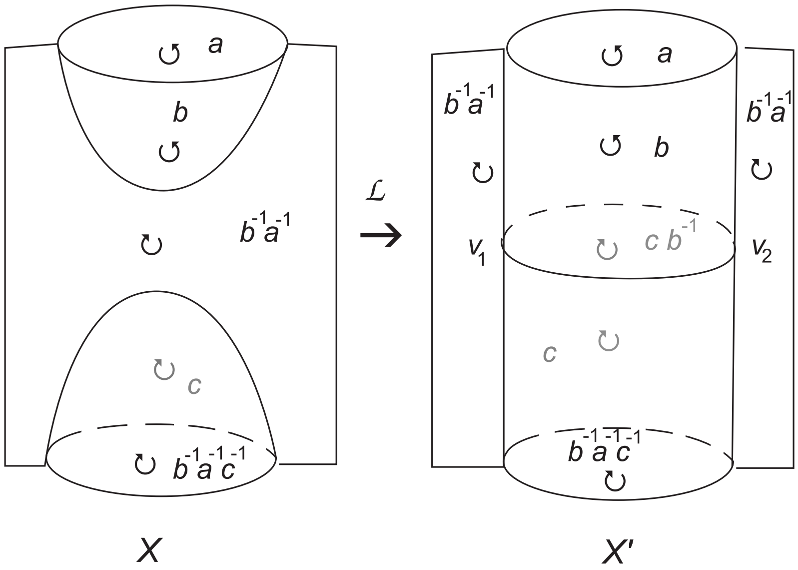


Fig. A.1.

Consider the vertices v_1 and v_2 of polyhedron X' (see Fig. A.1). Let t_1 and t_2 be the dual tetrahedra for v_1 and v_2 respectively. It is possible to order the vertices of t_1 and t_2 in such a way that $W^*(v_i, \varphi) = \alpha(c, a, b)^{\epsilon(t_i)}$, $i = 1, 2$. Notice that $\epsilon(t_1)$ and $\epsilon(t_1)$ in this case have to be of opposite signs. On the other hand, obviously, $\alpha(c, a, b)\alpha(c, a, b)^{-1} = 1$. This implies that $DW^*_{G,\alpha}(M)$ is invariant under the moves $\mathcal{L}^{\pm 1}$.

Let X' be a simple polyhedron obtained from X by a move $\mathcal{M}$ (see Fig. A.2). We may assume that the 2-components of X and X' are oriented and colored as in the picture.

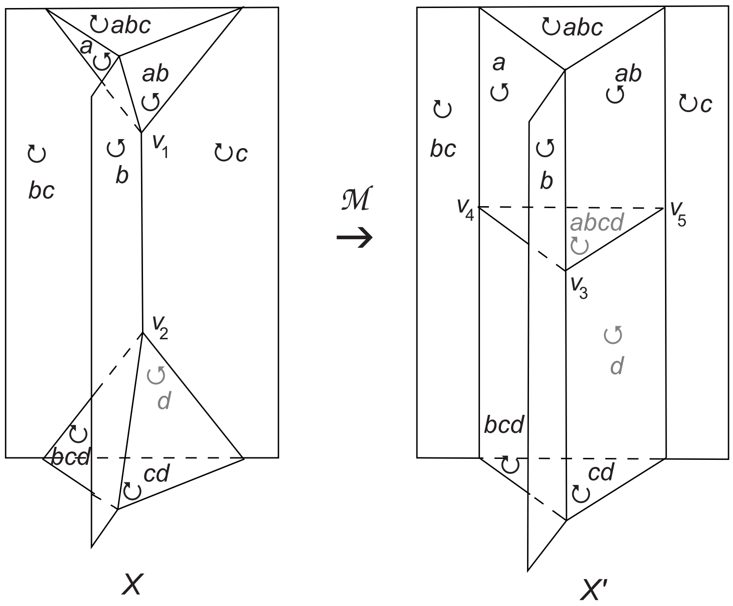


Fig. A.2.

Consider the vertices v_1, v_2, v_3, v_4, v_5 of the polyhedra X and X' (see Fig. A.2). Let t_1, t_2, t_3, t_4, t_5 be their respective dual tetrahedra. It is possible to order the vertices of t_1, t_2, t_3, t_4, t_5 in such a way that $W(v_1, \varphi) = \alpha(a, b, c), W(v_2, \varphi) = \alpha(b, c, d), W(v_3, \varphi) = \alpha(a, b, cd), W(v_4, \varphi) = \alpha(a, bc, d)^{-1}, W(v_5, \varphi) = \alpha(ab, c, d)$. Now invariance of $DW^*_{G,\alpha}(M)$ with respect to Matveev moves $\mathcal{M}^{\pm 1}$ follows from the following equation:

$$(\delta^3\alpha)(a, b, c, d) = \alpha(b, c, d)\alpha(ab, c, d)^{-1}\alpha(a, bc, d)\alpha(a, b, cd)^{-1}\alpha(a, b, c),$$

which is 1, because $\alpha \in H^3(G, \mathbf{Q}/\mathbf{Z})$.

Let X' be a simple polyhedron obtained from X by a move $\mathcal{B}$ (see Fig. A.3). Again, we may assume that the 2-components of X and X' are oriented and colored as in the picture.

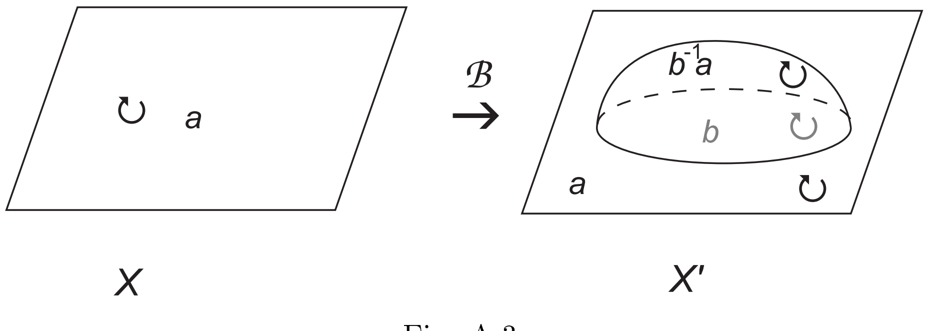


Fig. A.3.

Notice that if X is a simple spine of $M - (kB^3)$ then X' is a simple spine of $M - ((k+1)B^3)$. Also, X' has two extra components, one new 1-stratum, and no new vertices, therefore each admissible coloring of X corresponds to $|G|$ admissible colorings of X', having the same input into the sum for $DW^*_{G,\alpha}(M)$. Thus, invariance under the moves $\mathcal{B}^{\pm 1}$ is derived from the factor $\frac{1}{|G|^{k-1}}$ in the formula for $DW^*_{G,\alpha}(M)$.

2) Let T be a singular triangulation of M. Let X be the dual special 2-skeleton of M. Let n be the number of vertices of T. Then X is a special spine of $M - (nB^3)$. Thus, the number k in the definition of $DW^*_{G,\alpha}(M)$ coincides with the number n in the definition of $\widetilde{DW}_{G,\alpha}(M)$. The products $\prod_{t\in T}\widetilde{W}(t,\phi)$ and $\prod_{i=1}^{m} W^*(v_i,\varphi)$ coincide by definition. It remains to notice that by the 1st part of the theorem, $DW^*_{G,\alpha}(M)$ does not depend on the choice of a special spine X. This proves that $\widetilde{DW}_{G,\alpha}(M)$ does not depend on the choice of a singular triangulation, and thus is an invariant of closed oriented 3-manifolds. Furthermore, by Lemma A.3, it coincides with the invariant $DW_{G,\alpha}(M)$.

3) This follows from part 2. $\square$

APPENDIX B
TURAEV-VIRO INVARIANTS FOR IRREDUCIBLE CLOSED ORIENTABLE 3-MANIFOLDS OF COMPLEXITY ≤ 7

In this appendix we show values of the invariants TV_0, TV_1, TV_2 and TV^* from Chapter 1. The tables 1–135 contain the values of the invariants with the parameter $3 \leq r \leq 8$ for all irreducible closed orientable 3-manifolds of complexity ≤ 6. For a definition and the theory of 3-manifold complexity see [M2]. The computations were performed using the tables of special spines of the 3-manifolds of complexity ≤ 6 from [M3]. The tables 136–239 contain the values of the invariants with the parameter $3 \leq r \leq 7$ for all irreducible closed orientable 3-manifolds of complexity 7, excluding the lens spaces. The computations were performed using the tables of special spines of the 3-manifolds of complexity 7 from [Ov]. The computer encoding of the 3-manifolds of complexity 7 was made by an undergraduate student of the Chelyabinsk State University A. Gaaz.

Every number $TV_N(M)_q$, $N \in \{0, 1, 2\}$, is presented as a cyclotomic integer from $\mathbf{Q}(q)$, i.e. as an integer polynomial in q, where q is a $2r$th primitive root of unity. This distinguishes our tables from other tables of Turaev-Viro and Reshetikhin-Turaev invariants (see [KL1], [KL2], [N]), where the values where presented as complex numbers. Notice that the definition of the invariants only implies that $TV_N(M)_q, TV^*(M)_q \in \mathbf{Q}(q)$. However, it was proven (see [Mu1], [Mu2], [MR]) that Turaev-Viro invariants for prime r are cyclotomic integers. Our computations suggest that they are cyclotomic integers for composed values of r as well.

We also present the real values of the invariants $TV_N(M)_q$ and the real values of $TV^*(M)_q$ for $q = e^{i\pi/r}$. We do not explicitly present the values of the invariants parameterized by those q which are not primitive $2r$th roots of unity, but such that q^2 are primitive rth roots of unity. Such invariants can be obtained from the presented ones using the formula from Corollary 1.12.

Recall that the numbers in the tables are related by the following formula:

$$TV^*(M)_q = -\frac{(q-q^{-1})^2}{2r}(TV_0(M)_q + TV_1(M)_q + TV_2(M)_q).$$

Now, let us explain the chosen notations for the manifolds. The 3-sphere, the projective space, and the lens spaces are denoted as usual by S^3, $\mathbf{R}P^3$, and $L_{p,q}$, respectively. A manifold S^3/G is a quotient space of S^3 by a certain free action of the finite group G. The list of the groups can be found in [M3], [Ov], and [Fo]. Let

M_{2T} be a 3-manifold, possibly not connected, with the boundary consisting of two tori. By

$$\frac{M_{2T}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

we denote the 3-manifold obtained by gluing together the tori using the homeomorphism defined by the matrix in the denominator. Finally, $ZF(e)$ denotes an orientable Seifert 3-manifold with the base S^2, three special fibers of the type $(3,1)$, and the Euler number e. For most of the manifolds of complexity 7 we also present their Seifert structure (see [Ov]). For more details on the presented 3-manifolds see [M3], [F2], [Ov], [MOS].

All the computations were made using a Turbo Pascal computer program developed by the author during his work and study in the Chelyabinsk State University in 1992–1996. The tables with the manifolds of complexity ≤ 3 and with $r \leq 7$ were first published in [S1].

Manifolds of complexity 0

Table 1: Invariants for S^3

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	0 =0.000	0.500
4	1 =1.000	0 =0.000	0 =0.000	0.250
5	1 =1.000	0 =0.000	0 =0.000	0.138
6	1 =1.000	0 =0.000	0 =0.000	0.083
7	1 =1.000	0 =0.000	0 =0.000	0.054
8	1 =1.000	0 =0.000	0 =0.000	0.037

Table 2: Invariants for $\mathbf{R}P^3$

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	-1 =−1.000	0 =0.000	0.000
4	2 =2.000	q^3-q =−1.414	0 =0.000	0.146
5	$-q^3+q^2+2$ =2.618	q^3-q^2-2 =−2.618	0 =0.000	0.000
6	4 =4.000	$2q^3-4q$ =−3.464	0 =0.000	0.045
7	$-2q^5+q^4-q^3+2q^2+3$ =5.049	$2q^5-q^4+q^3-2q^2-3$ =−5.049	0 =0.000	0.000
8	$-2q^6+2q^2+4$ =6.828	$3q^7+q^5-q^3-3q^1$ =−6.309	0 =0.000	0.019

Table 3: Invariants for $L_{3,1}$

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	0 =0.000	0.500
4	1 =1.000	0 =0.000	0 =0.000	0.250
5	$-q^3+q^2+2$ =2.618	0 =0.000	0 =0.000	0.362
6	3 =3.000	0 =0.000	0 =0.000	0.250
7	$-q^5+q^2+2$ =3.247	0 =0.000	0 =0.000	0.175
8	$-2q^6+2q^2+3$ =5.828	0 =0.000	0 =0.000	0.213

Manifolds of complexity 1

Table 4: Invariants for $L_{4,1}$

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	1 =1.000	1.000
4	2 =2.000	0 =0.000	0 =0.000	0.500
5	1 =1.000	0 =0.000	1 =1.000	0.276
6	4 =4.000	0 =0.000	0 =0.000	0.333
7	$-q^5+q^2+2$ =3.247	0 =0.000	$-q^5+q^2+2$ =3.247	0.349
8	$4+2q^2-2q^6$ =6.828	0 =0.000	0 =0.000	0.250

Table 5: Invariants for $L_{5,2}$

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	0 =0.000	0.500
4	1 =1.000	0 =0.000	0 =0.000	0.250
5	0 =0.000	0 =0.000	0 =0.000	0.000
6	1 =1.000	0 =0.000	0 =0.000	0.083
7	$-2q^5+q^4-q^3+2q^2+3$ =5.049	0 =0.000	0 =0.000	0.272
8	$3+2q^2-2q^6$ =5.828	0 =0.000	0 =0.000	0.213

Manifolds of complexity 2

Table 6: Invariants for $L_{5,1}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$-q^3+q^2+3$	=3.618	0	=0.000	0	=0.000	0.500
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$-2q^5+q^4-q^3+2q^2+3$	=5.049	0	=0.000	0	=0.000	0.272
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 7: Invariants for $L_{7,2}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$-q^3+q^2+2$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	0	=0.000	0	=0.000	0	=0.000	0.000
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 8: Invariants for $L_{8,3}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$-q^3+q^2+2$	=2.618	0	=0.000	$-q^3+q^2+2$	=2.618	0.724
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	1	=1.000	0	=0.000	1	=1.000	0.108
8	0	=0.000	0	=0.000	0	=0.000	0.000

Table 9: Invariants for S^3/Q_8

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	3	=3.000	2.000
4	4	=4.000	0	=0.000	6	=6.000	2.500
5	$-q^3+q^2+4$	=4.618	0	=0.000	$-3q^3+3q^2+12$	=13.854	2.553
6	10	=10.000	0	=0.000	18	=18.000	2.333
7	$-2q^5+2q^2+7$	=9.494	0	=0.000	$-6q^5+6q^2+21$	=28.482	2.043
8	$12+4q^2-4q^6$	=17.657	0	=0.000	$24+6q^2-6q^6$	=32.485	1.836

Manifolds of complexity 3

Table 10: Invariants for $L_{6,1}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$-q^3+q$	=1.414	0	=0.000	0.853
5	1	=1.000	-1	=−1.000	0	=0.000	0.000
6	6	=6.000	0	=0.000	0	=0.000	0.500
7	1	=1.000	-1	=−1.000	0	=0.000	0.000
8	$4+2q^2-2q^6$	=6.828	$q^1+q^3-q^5-q^7$	=2.613	0	=0.000	0.346

Table 11: Invariants for $L_{9,2}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$-2q^5+q^4-q^3+2q^2+3$	=5.049	0	=0.000	0	=0.000	0.272
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 12: Invariants for $L_{10,3}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$-q^3+q$	=1.414	0	=0.000	0.853
5	0	=0.000	0	=0.000	0	=0.000	0.000
6	4	=4.000	$-2q^3+4q$	=3.464	0	=0.000	0.622
7	$-q^5+q^2+2$	=3.247	q^5-q^2-2	=−3.247	0	=0.000	0.000
8	$4+2q^2-2q^6$	=6.828	$-q^1-q^3+q^5+q^7$	=−2.613	0	=0.000	0.154

Table 13: Invariants for $L_{11,4}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$-q^5+q^2+2$	=3.247	0	=0.000	0	=0.000	0.175
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 14: Invariants for $L_{12,5}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$-q^3+q^2+2$	=2.618	0	=0.000	$-q^3+q^2+2$	=2.618	0.724
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$-2q^5+q^4-q^3+2q^2+3$	=5.049	0	=0.000	$-2q^5+q^4-q^3+2q^2+3$	=5.049	0.543
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 15: Invariants for $L_{13,5}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$-q^3+q^2+2$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 16: Invariants for S^3/Q_{12}

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$-q^3+q^2+4$	=4.618	0	=0.000	$-q^3+q^2+4$	=4.618	1.276
6	10	=10.000	0	=0.000	6	=6.000	1.333
7	$-2q^5+q^4-q^3+2q^2+5$	= 7.049	0	=0.000	$-2q^5+q^4-q^3+2q^2+5$	=7.049	0.758
8	$12+6q^2-6q^6$	=20.485	0	=0.000	$8+4q^2-4q^6$	=13.657	1.250

Manifolds of complexity 4

Table 17: Invariants for $L_{7,1}$*:*

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	0	=0.000	0	=0.000	0	=0.000	0.000
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	0	=0.000	0	=0.000	0.500
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 18: Invariants for $L_{11,2}$*:*

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$2+q^2-q^5$	=3.247	0	=0.000	0	=0.000	0.175
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 19: Invariants for $L_{13,3}$*:*

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 20: Invariants for $L_{14,3}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
6	4	$=4.000$	$4q^1-2q^3$	$=3.464$	0	$=0.000$	0.622
7	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$3q^1+q^3-q^5-3q^7$	$=6.309$	0	$=0.000$	0.481

Table 21: Invariants for $L_{15,4}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$3+q^2-q^3$	$=3.618$	0	$=0.000$	0	$=0.000$	0.500
6	3	$=3.000$	0	$=0.000$	0	$=0.000$	0.250
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 22: Invariants for $L_{16,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	2	$=2.000$	1.000
5	1	$=1.000$	$0=0.000$	1	$=1.000$	0.276
6	4	$=4.000$	$0=0.000$	0	$=0.000$	0.333
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	$0=0.000$	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0.543
8	$8+4q^2-4q^6$	$=13.657$	$0=0.000$	$8+4q^2-4q^6$	$=13.657$	1.000

Table 23: Invariants for $L_{17,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 24: Invariants for $L_{18,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	$2+q^2-q^3$	$=2.618$	$-2-q^2+q^3$	$=-2.618$	0	$=0.000$	0.000
6	6	$=6.000$	0	$=0.000$	0	$=0.000$	0.500
7	$2+q^2-q^5$	$=3.247$	$-2-q^2+q^5$	$=-3.247$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$3q^1+q^3-q^5-3q^7$	$=6.309$	0	$=0.000$	0.481

Table 25: Invariants for $L_{19,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 26: Invariants for $L_{21,8}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	0	=0.000	0	=0.000	0.500
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 27: Invariants for S^3/D_{24}:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$5+2q^2-2q^3$	=6.236	0 =0.000	$5+2q^2-2q^3$	=6.236	1.724
6	10	=10.000	0 =0.000	6	=6.000	1.333
7	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	0 =0.000	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	1.436
8	$16+8q^2-8q^6$	=27.314	0 =0.000	$16+8q^2-8q^6$	=27.314	2.000

Table 28: Invariants for $S^3/Q_8 \times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$5+2q^2-2q^3$	=6.236	0 =0.000	$15+6q^2-6q^3$	=18.708	3.447
6	12	=12.000	0 =0.000	36	=36.000	4.000
7	$12+7q^2-2q^3+2q^4-7q^5$	=19.839	0 =0.000	$36+21q^2-6q^3+6q^4-21q^5$	=59.516	4.268
8	$20+12q^2-12q^6$	=36.971	0 =0.000	$48+30q^2-30q^6$	=90.426	4.664

Table 29: Invariants for S^3/Q_{16}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	=−2.000	1	=1.000	0.000
4	4	=4.000	0	=0.000	2	=2.000	1.500
5	1	=1.000	-2	=−2.000	1	=1.000	0.000
6	10	=10.000	$8q^1-4q^3$	=6.928	6	=6.000	1.911
7	$5+2q^2-q^3+q^4-2q^5$	=7.049	$-10-4q^2+2q^3-2q^4+4q^5$	=−14.098	$5+2q^2-q^3+q^4-2q^5$	=7.049	0.000
8	$16+8q^2-8q^6$	=27.314	$4q^1+4q^3-4q^5-4q^7$	=10.453	$12+6q^2-6q^6$	=20.485	2.133

Table 30: Invariants for S^3/P_{24}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	0	=0.000	0	=0.000	0.543
8	$11+6q^2-6q^6$	=19.485	0	=0.000	0	=0.000	0.713

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Table 31: Invariants for $L_{8,1}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	0	=0.000	0	=0.000	0	=0.000	0.000
4	2	=2.000	0	=0.000	-2	=−2.000	0.000
5	$2+q^2-q^3$	=2.618	0	=0.000	$2+q^2-q^3$	=2.618	0.724
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	1	=1.000	0	=0.000	1	=1.000	0.108
8	0	=0.000	0	=0.000	0	=0.000	0.000

Table 32: Invariants for $L_{13,6}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 33: Invariants for $L_{20,9}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$3+q^2-q^3$	=3.618	0	=0.000	$3+q^2-q^3$	=3.618	1.000
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	1	=1.000	0	=0.000	1	=1.000	0.108
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 34: Invariants for $L_{16,3}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	-2	=−2.000	0.000
5	1	=1.000	0 =0.000	1	=1.000	0.276
6	4	=4.000	0 =0.000	0	=0.000	0.333
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0 =0.000	$3+2q^2-q^3+q^4-2q^5$	=5.049	0.543
8	0	=0.000	0	=0.000	0 =0.000	0.000

Table 35: Invariants for $L_{17,3}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 36: Invariants for $L_{25,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$3+q^2-q^3$	$=3.618$	0	$=0.000$	0	$=0.000$	0.500
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 37: Invariants for $L_{19,4}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0	$=0.000$	0	$=0.000$	0.272
8	$3+2q^2-2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 38: Invariants for $L_{26,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	$=1.000$	-1	$=-1.000$	0 $=0.000$	0.000
4	2	$=2.000$	q^1-q^3	$=1.414$	0 $=0.000$	0.854
5	1	$=1.000$	-1	$=-1.000$	0 $=0.000$	0.000
6	4	$=4.000$	$-4q^1+2q^3$	$=-3.464$	0 $=0.000$	0.045
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	$-3-2q^2+q^3-q^4+2q^5$	$=-5.049$	0 $=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$q^1+q^3-q^5-q^7$	$=2.613$	0 $=0.000$	0.346

Table 39: Invariants for $L_{24,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	-2	$=-2.000$	0.000
5	1	$=1.000$	0	$=0.000$	1	$=1.000$	0.276
6	6	$=6.000$	0	$=0.000$	-6	$=-6.000$	0.000
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	$2+q^2-q^5$	$=3.247$	0.349
8	$8+4q^2-4q^6$	$=13.657$	0	$=0.000$	0	$=0.000$	0.500

Table 40: Invariants for $L_{27,10}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	3	$=3.000$	0	$=0.000$	0	$=0.000$	0.250
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	$3+2q^2-2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 41: Invariants for $L_{23,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0	$=0.000$	0	$=0.000$	0.272
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 42: Invariants for $L_{31,13}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 43: Invariants for $L_{29,8}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	$3+2q^2-2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 44: Invariants for $L_{34,13}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
6	4	$=4.000$	$4q^1-2q^3$	$=3.464$	0	$=0.000$	0.622
7	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$-3q^1-q^3+q^5+3q^7$	$=-6.309$	0	$=0.000$	0.019

Table 45: Invariants for $L_{22,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$q^1 - q^3$	$=1.414$	0	$=0.000$	0.854
5	$2 + q^2 - q^3$	$=2.618$	$-2 - q^2 + q^3$	$=-2.618$	0	$=0.000$	0.000
6	4	$=4.000$	$-4q^1 + 2q^3$	$=-3.464$	0	$=0.000$	0.045
7	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
8	$4 + 2q^2 - 2q^6$	$=6.828$	$-q^1 - q^3 + q^5 + q^7$	$=-2.613$	0	$=0.000$	0.154

Table 46: Invariants for $L_{29,12}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	$3 + 2q^2 - 2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 47: Invariants for $L_{17,13}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2 + q^2 - q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2 + q^2 - q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 48: Invariants for $L_{25,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2 + q^2 - q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 49: Invariants for $L_{30,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	$=1.000$	-1	$=-1.000$	0 $=0.000$	0.000
4	2	$=2.000$	$-q^1 + q^3$	$=-1.414$	0 $=0.000$	0.146
5	$3 + q^2 - q^3$	$=3.618$	$-3 - q^2 + q^3$	$=-3.618$	0 $=0.000$	0.000
6	6	$=6.000$	0	$=0.000$	0 $=0.000$	0.500
7	$3 + 2q^2 - q^3 + q^4 - 2q^5$	$=5.049$	$-3 - 2q^2 + q^3 - q^4 + 2q^5$	$=-5.049$	0 $=0.000$	0.000
8	$4 + 2q^2 - 2q^6$	$=6.828$	$-3q^1 - q^3 + q^5 + 3q^7$	$=-6.309$	0 $=0.000$	0.019

Table 50: Invariants for $L_{23,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 51: Invariants for S^3/P_{48}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=-1.000	0	=0.000	0.000
4	2	=2.000	q^1-q^3	=1.414	0	=0.000	0.854
5	$2+q^2-q^3$	=2.618	$-2-q^2+q^3$	=-2.618	0	=0.000	0.000
6	10	=10.000	0	=0.000	0	=0.000	0.833
7	2	=2.000	-2	=-2.000	0	=0.000	0.000
8	$12+6q^2-6q^6$	=20.485	$q^1+q^3-q^5-q^7$	=2.613	0	=0.000	0.846

Table 52: Invariants for S^3/P'_{72}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	2	=2.000	0	=0.000	0	=0.000	0.108
8	$9+4q^2-4q^6$	=14.657	0	=0.000	0	=0.000	0.537

Table 53: Invariants for S^3/Q_{20}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$5+3q^2-3q^3$	=6.854	0	=0.000	$5+3q^2-3q^3$	=6.854	1.894
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$7+2q^2-2q^5$	=9.494	0	=0.000	$7+2q^2-2q^5$	=9.494	1.021
8	$12+6q^2-6q^6$	=20.485	0	=0.000	$8+4q^2-4q^6$	=13.657	1.250

Table 54: Invariants for S^3/D_{40}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$1-q^2+q^3$	=0.382	0	=0.000	$1-q^2+q^3$	=0.382	0.106
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	0	=0.000	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	1.845
8	$16+8q^2-8q^6$	=27.314	0	=0.000	$16+8q^2-8q^6$	=27.314	2.000

Table 55: Invariants for $S^3/Q_{16} \times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	0	$=0.000$	2	$=2.000$	1.500
5	$2+q^2-q^3$	$=2.618$	$-4-2q^2+2q^3$	$=-5.236$	$2+q^2-q^3$	$=2.618$	0.000
6	12	$=12.000$	0	$=0.000$	12	$=12.000$	2.000
7	$5+2q^2-2q^5$	$=7.494$	$-10-4q^2+4q^5$	$=-14.988$	$5+2q^2-2q^5$	$=7.494$	0.000
8	$16+8q^2-8q^6$	$=27.314$	$12q^1+4q^3-4q^5-12q^7$	$=25.235$	$12+6q^2-6q^6$	$=20.485$	2.674

Table 56: Invariants for $S^3/Q_{20} \times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	2	$=2.000$	1.000
5	$1-q^2+q^3$	$=0.382$	$0=0.000$	$1-q^2+q^3$	$=0.382$	0.106
6	4	$=4.000$	$0=0.000$	0	$=0.000$	0.333
7	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	$0=0.000$	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	1.845
8	$16+8q^2-8q^6$	$=27.314$	$0=0.000$	$16+8q^2-8q^6$	$=27.314$	2.000

Table 57: Invariants for S^3/D_{48}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	$4+q^2-q^3$	$=4.618$	0	$=0.000$	$4+q^2-q^3$	$=4.618$	1.276
6	10	$=10.000$	0	$=0.000$	6	$=6.000$	1.333
7	$5+2q^2-2q^5$	$=7.494$	0	$=0.000$	$5+2q^2-2q^5$	$=7.494$	0.806
8	$8+4q^2-4q^6$	$=13.657$	0	$=0.000$	$8+4q^2-4q^6$	$=13.657$	1.000

Table 58: Invariants for $S^3/Q_8 \times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	$0=0.000$	6	$=6.000$	2.500
5	$3+q^2-q^3$	$=3.618$	$0=0.000$	$9+3q^2-3q^3$	$=10.854$	2.000
6	10	$=10.000$	$0=0.000$	18	$=18.000$	2.333
7	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	$0=0.000$	$33+18q^2-9q^3+9q^4-18q^5$	$=51.440$	3.689
8	$20+12q^2-12q^6$	$=36.971$	$0=0.000$	$48+30q^2-30q^6$	$=90.426$	4.664

Table 59: Invariants for $S^3/Q_{12} \times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$3+q^2-q^3$	$=3.618$	$0=0.000$	$3+q^2-q^3$	$=3.618$	1.000
6	10	$=10.000$	$0=0.000$	6	$=6.000$	1.333
7	$8+5q^2-2q^3+2q^4-5q^5$	$=13.345$	$0=0.000$	$8+5q^2-2q^3+2q^4-5q^5$	$=13.345$	1.436
8	$12+6q^2-6q^6$	$=20.485$	$0=0.000$	$8+4q^2-4q^6$	$=13.657$	1.250

Table 60: Invariants for $S^3/P_{24}\times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$4+2q^2-2q^5$	=6.494	0	=0.000	0	=0.000	0.349
8	$9+4q^2-4q^6$	=14.657	0	=0.000	0	=0.000	0.537

Table 61: Invariants for S^3/P_{120}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+3q^2-3q^3$	=6.854	0	=0.000	0	=0.000	0.947
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	0	=0.000	0	=0.000	0.718
8	$9+4q^2-4q^6$	=14.657	0	=0.000	0	=0.000	0.537

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Table 62: Invariants for $L_{9,1}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	0	=0.000	0	=0.000	0	=0.000	0.000
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 63: Invariants for $L_{15,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	0	=0.000	0	=0.000	0	=0.000	0.000
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 64: Invariants for $L_{19,6}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 65: Invariants for $L_{20,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	0	=0.000	0	=0.000	0	=0.000	0.000
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	1	=1.000	0	=0.000	1	=1.000	0.108
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 66: Invariants for $L_{29,9}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 67: Invariants for $L_{31,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$2+q^2-q^5$	=3.247	0	=0.000	0	=0.000	0.175
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 68: Invariants for $L_{36,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	1	=1.000	0	=0.000	1	=1.000	0.276
6	6	=6.000	0	=0.000	-6	$=-6.000$	0.000
7	1	=1.000	0	=0.000	1	=1.000	0.108
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 69: Invariants for $L_{39,14}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$2+q^2-q^5$	=3.247	0	=0.000	0	=0.000	0.175
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 70: Invariants for $L_{23,4}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0	$=0.000$	0	$=0.000$	0.272
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 71: Invariants for $L_{34,15}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
6	4	$=4.000$	$4q^1-2q^3$	$=3.464$	0	$=0.000$	0.622
7	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$-3q^1-q^3+q^5+3q^7$	$=-6.309$	0	$=0.000$	0.019

Table 72: Invariants for $L_{37,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0	$=0.000$	0	$=0.000$	0.272
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 73: Invariants for $L_{41,15}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 74: Invariants for $L_{28,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	0	$=0.000$	0.500
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	$2+q^2-q^3$	$=2.618$	0.724
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	0	$=0.000$	0	$=0.000$	0.250

Table 75: Invariants for $L_{41,18}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.500
4	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.250
5	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.138
6	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.083
7	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.054
8	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.037

Table 76: Invariants for $L_{44,13}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	$1 \quad =1.000$	$0 =0.000$	$1 \quad =1.000$	1.000
4	$2 \quad =2.000$	$0 =0.000$	$0 \quad =0.000$	0.500
5	$1 \quad =1.000$	$0 =0.000$	$1 \quad =1.000$	0.276
6	$4 \quad =4.000$	$0 =0.000$	$0 \quad =0.000$	0.333
7	$3+2q^2-q^3+q^4-2q^5 \quad =5.049$	$0 =0.000$	$3+2q^2-q^3+q^4-2q^5 \quad =5.049$	0.543
8	$4+2q^2-2q^6 \quad =6.828$	$0 =0.000$	$0 \quad =0.000$	0.250

Table 77: Invariants for $L_{49,19}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.500
4	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.250
5	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.138
6	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.083
7	$0 \quad =0.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.000
8	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.037

Table 78: Invariants for $L_{37,8}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.500
4	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.250
5	$2+q^2-q^3 \quad =2.618$	$0 \quad =0.000$	$0 \quad =0.000$	0.362
6	$1 \quad =1.000$	$0 \quad =0.000$	$0 \quad =0.000$	0.083
7	$3+2q^2-q^3+q^4-2q^5 \quad =5.049$	$0 \quad =0.000$	$0 \quad =0.000$	0.272
8	$3+2q^2-2q^6 \quad =5.828$	$0 \quad =0.000$	$0 \quad =0.000$	0.213

Table 79: Invariants for $L_{50,21}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	$1 \quad =1.000$	$-1 \quad =-1.000$	$0 \quad =0.000$	0.000
4	$2 \quad =2.000$	$-q^1+q^3 \quad =-1.414$	$0 \quad =0.000$	0.146
5	$3+q^2-q^3 \quad =3.618$	$-3-q^2+q^3 \quad =-3.618$	$0 \quad =0.000$	0.000
6	$4 \quad =4.000$	$-4q^1+2q^3 \quad =-3.464$	$0 \quad =0.000$	0.045
7	$1 \quad =1.000$	$-1 \quad =-1.000$	$0 \quad =0.000$	0.000
8	$4+2q^2-2q^6 \quad =6.828$	$3q^1+q^3-q^5-3q^7 \quad =6.309$	$0 \quad =0.000$	0.481

Table 80: Invariants for $L_{47,13}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 81: Invariants for $L_{55,21}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 82: Invariants for $L_{27,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 83: Invariants for $L_{39,17}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$2+q^2-q^5$	=3.247	0	=0.000	0	=0.000	0.175
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 84: Invariants for $L_{41,12}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	1	=1.000	0	=0.000	0	=0.000	0.054
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 85: Invariants for $L_{46,17}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
6	4	$=4.000$	$-4q^1+2q^3$	$=-3.464$	0	$=0.000$	0.045
7	$2+q^2-q^5$	$=3.247$	$-2-q^2+q^5$	$=-3.247$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$3q^1+q^3-q^5-3q^7$	$=6.309$	0	$=0.000$	0.481

Table 86: Invariants for $L_{30,13}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000
6	6	$=6.000$	0	$=0.000$	0	$=0.000$	0.500
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	$-3-2q^2+q^3-q^4+2q^5$	$=-5.049$	0	$=0.000$	0.000
8	$4+2q^2-2q^6$	$=6.828$	$-3q^1-q^3+q^5+3q^7$	$=-6.309$	0	$=0.000$	0.019

Table 87: Invariants for $L_{21,4}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	3	$=3.000$	0	$=0.000$	0	$=0.000$	0.250
7	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000
8	$3+2q^2-2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 88: Invariants for $L_{24,5}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	1	$=1.000$	0	$=0.000$	1	$=1.000$	0.276
6	6	$=6.000$	0	$=0.000$	-6	$=-6.000$	0.000
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	$2+q^2-q^5$	$=3.247$	0.349
8	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000

Table 89: Invariants for $L_{33,14}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	3	$=3.000$	0	$=0.000$	0	$=0.000$	0.250
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0	$=0.000$	0	$=0.000$	0.272
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 90: Invariants for $L_{24,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	1	=1.000	0	=0.000	1	=1.000	0.276
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$2+q^2-q^5$	=3.247	0	=0.000	$2+q^2-q^5$	=3.247	0.349
8	0	=0.000	0	=0.000	0	=0.000	0.000

Table 91: Invariants for $L_{45,19}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$2+q^2-q^5$	=3.247	0	=0.000	0	=0.000	0.175
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 92: Invariants for $L_{40,11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$3+q^2-q^3$	=3.618	0 =0.000	$3+q^2-q^3$	=3.618	1.000
6	4	=4.000	0 =0.000	0	=0.000	0.333
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0 =0.000	$3+2q^2-q^3+q^4-2q^5$	=5.049	0.543
8	0	=0.000	0 =0.000	0	=0.000	0.000

Table 93: Invariants for $L_{35,8}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	0	=0.000	0	=0.000	0	=0.000	0.000
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	0	=0.000	0	=0.000	0.500
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 94: Invariants for $L_{33,10}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	0	=0.000	0	=0.000	0.272
8	1	=1.000	0	=0.000	0	=0.000	0.037

Table 95: Invariants for $L_{32,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	$2+q^2-q^3$	$=2.618$	0.724
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	$2+q^2-q^5$	$=3.247$	0.349
8	$8+4q^2-4q^6$	$=13.657$	0	$=0.000$	$-8-4q^2+4q^6$	$=-13.657$	0.000

Table 96: Invariants for $L_{43,18}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	0	$=0.000$	0.362
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.054
8	$3+2q^2-2q^6$	$=5.828$	0	$=0.000$	0	$=0.000$	0.213

Table 97: Invariants for $L_{31,7}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.138
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	0	$=0.000$	0.175
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 98: Invariants for $\dfrac{T\times I}{\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	$0=0.000$	0	$=0.000$	1.000
5	$2-q^2+q^3$	$=1.382$	$0=0.000$	$6-3q^2+3q^3$	$=4.146$	0.764
6	18	$=18.000$	$0=0.000$	18	$=18.000$	3.000
7	$12+6q^2-2q^3+2q^4-6q^5$	$=18.592$	$0=0.000$	$36+18q^2-6q^3+6q^4-18q^5$	$=55.775$	4.000
8	$24+8q^2-8q^6$	$=35.314$	$0=0.000$	24	$=24.000$	2.172

Table 99: Invariants for $\frac{T\times I}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	=-2.000	1	=1.000	0.000
4	4	=4.000	0	=0.000	0	=0.000	1.000
5	0	=0.000	0	=0.000	0	=0.000	0.000
6	12	=12.000	0	=0.000	0	=0.000	1.000
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	$-12-6q^2+2q^3-2q^4+6q^5$	=-18.592	$6+3q^2-q^3+q^4-3q^5$	=9.296	0.000
8	$16+8q^2-8q^6$	=27.314	0	=0.000	0	=0.000	1.000

Table 100: Invariants for $\frac{T\times I}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	0	=0.000	1.000
5	$2-q^2+q^3$	=1.382	0 =0.000	$6-3q^2+3q^3$	=4.146	0.764
6	18	=18.000	0 =0.000	18	=18.000	3.000
7	$12+6q^2-2q^3+2q^4-6q^5$	=18.592	0 =0.000	$36+18q^2-6q^3+6q^4-18q^5$	=55.775	4.000
8	$24+8q^2-8q^6$	=35.314	0 =0.000	24	=24.000	2.172

Table 101: Invariants for $\frac{T\times I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	7	=7.000	4.000
4	8	=8.000	0 =0.000	28	=28.000	9.000
5	$12+4q^2-4q^3$	=14.472	0 =0.000	$84+28q^2-28q^3$	=101.305	16.000
6	48	=48.000	0 =0.000	252	=252.000	25.000
7	$54+27q^2-9q^3+9q^4-27q^5$	=83.663	0 =0.000	$378+189q^2-63q^3+63q^4-189q^5$	=585.642	36.000
8	$112+56q^2-56q^6$	=191.196	0 =0.000	$672+336q^2-336q^6$	=1147.176	49.000

Table 102: Invariants for $\frac{T\times I}{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	-2	=-2.000	0.000
5	$3+q^2-q^3$	=3.618	0	=0.000	$3+q^2-q^3$	=3.618	1.000
6	12	=12.000	0	=0.000	0	=0.000	1.000
7	0	=0.000	0	=0.000	0	=0.000	0.000
8	$16+8q^2-8q^6$	=27.314	0	=0.000	0	=0.000	1.000

Table 103: Invariants for $\frac{T\times I}{\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	1 =1.000	1.000
4	2 =2.000	0 =0.000	-2 =-2.000	0.000
5	$3+q^2-q^3$ =3.618	0 =0.000	$3+q^2-q^3$ =3.618	1.000
6	12 =12.000	0 =0.000	0 =0.000	1.000
7	0 =0.000	0 =0.000	0 =0.000	0.000
8	$16+8q^2-8q^6$ =27.314	0 =0.000	0 =0.000	1.000

Table 104: Invariants for $\frac{T\times I}{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	7 =7.000	4.000
4	8 =8.000	0 =0.000	28 =28.000	9.000
5	$12+4q^2-4q^3$ =14.472	0 =0.000	$84+28q^2-28q^3$ =101.305	16.000
6	48 =48.000	0 =0.000	252 =252.000	25.000
7	$54+27q^2-9q^3+9q^4-27q^5$ =83.663	0 =0.000	$378+189q^2-63q^3+63q^4-189q^5$ =585.642	36.000
8	$112+56q^2-56q^6$ =191.196	0 =0.000	$672+336q^2-336q^6$ =1147.176	49.000

Table 105: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	3 =3.000	2.000
4	4 =4.000	0 =0.000	12 =12.000	4.000
5	$8+4q^2-4q^3$ =10.472	0 =0.000	$24+12q^2-12q^3$ =31.416	5.789
6	28 =28.000	0 =0.000	72 =72.000	8.333
7	$30+19q^2-9q^3+9q^4-19q^5$ =49.687	0 =0.000	$90+57q^2-27q^3+27q^4-57q^5$ =149.062	10.690
8	$48+32q^2-32q^6$ =93.255	0 =0.000	$144+96q^2-96q^6$ =279.765	13.657

Table 106: Invariants for $S^3/P_{120}\times\mathbf{Z}_7$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	0 =0.000	0.500
4	1 =1.000	0 =0.000	0 =0.000	0.250
5	$1-q^2+q^3$ =0.382	0 =0.000	0 =0.000	0.053
6	1 =1.000	0 =0.000	0 =0.000	0.083
7	$6+3q^2-q^3+q^4-3q^5$ =9.296	0 =0.000	0 =0.000	0.500
8	$9+4q^2-4q^6$ =14.657	0 =0.000	0 =0.000	0.537

Table 107: Invariants for S^3/D_{56}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$5+2q^2-2q^3$	=6.236	0	=0.000	$5+2q^2-2q^3$	=6.236	1.724
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$2-q^2+3q^3-3q^4+q^5$	=2.088	0	=0.000	$2-q^2+3q^3-3q^4+q^5$	=2.088	0.225
8	0	=0.000	0	=0.000	0	=0.000	0.000

Table 108: Invariants for $S^3/P_{48}\times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	=1.000	-1	=−1.000	0 =0.000	0.000
4	2	=2.000	$-q^1+q^3$	=−1.414	0 =0.000	0.146
5	$3+q^2-q^3$	=3.618	$-3-q^2+q^3$	=−3.618	0 =0.000	0.000
6	10	=10.000	0	=0.000	0 =0.000	0.833
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	$-6-4q^2+2q^3-2q^4+4q^5$	=−10.098	0 =0.000	0.000
8	$12+6q^2-6q^6$	=20.485	$-3q^1-q^3+q^5+3q^7$	=−6.309	0 =0.000	0.519

Table 109: Invariants for S^3/Q_{24}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	3	=3.000	2.000
4	4	=4.000	0	=0.000	-2	=−2.000	0.500
5	1	=1.000	0	=0.000	3	=3.000	0.553
6	16	=16.000	0	=0.000	12	=12.000	2.333
7	$2+q^2-q^5$	=3.247	0	=0.000	$6+3q^2-3q^5$	=9.741	0.699
8	$12+4q^2-4q^6$	=17.657	0	=0.000	$-2q^2+2q^6$	=−2.828	0.543

Table 110: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}1&0\\1&1\end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	7	=7.000	4.000
4	8	=8.000	0 =0.000	24	=24.000	8.000
5	$10+3q^2-3q^3$	=11.854	0 =0.000	$70+21q^2-21q^3$	=82.979	13.106
6	46	=46.000	0 =0.000	198	=198.000	20.333
7	$44+22q^2-6q^3+6q^4-22q^5$	=68.763	0 =0.000	$308+154q^2-42q^3+42q^4-154q^5$	=481.343	29.589
8	$96+48q^2-48q^6$	=163.882	0 =0.000	$544+272q^2-272q^6$	=928.666	40.000

Table 111: Invariants for $S^3/Q_{28}\times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$5+2q^2-2q^3$	=6.236	0	=0.000	$5+2q^2-2q^3$	=6.236	1.724
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$2-q^2-q^3+q^4+q^5$	=0.308	0	=0.000	$2-q^2-q^3+q^4+q^5$	=0.308	0.033
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 112: Invariants for $S^3/Q_{32} \times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	$-4q^1+4q^3$	$=-5.657$	2	$=2.000$	0.086
5	$5+2q^2-2q^3$	$=6.236$	$-10-4q^2+4q^3$	$=-12.472$	$5+2q^2-2q^3$	$=6.236$	0.000
6	12	$=12.000$	0	$=0.000$	12	$=12.000$	2.000
7	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	$-6-4q^2+2q^3-2q^4+4q^5$	$=-10.098$	$3+2q^2-q^3+q^4-2q^5$	$=5.049$	0.000
8	8	$=8.000$	0	$=0.000$	$4-2q^2+2q^6$	$=1.172$	0.336

Table 113: Invariants for $S^3/P_{48} \times \mathbf{Z}_7$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	$=1.000$	-1	$=-1.000$	$0=0.000$	0.000
4	2	$=2.000$	q^1-q^3	$=1.414$	$0=0.000$	0.854
5	1	$=1.000$	-1	$=-1.000$	$0=0.000$	0.000
6	10	$=10.000$	0	$=0.000$	$0=0.000$	0.833
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$-6-3q^2+q^3-q^4+3q^5$	$=-9.296$	$0=0.000$	0.000
8	$12+6q^2-6q^6$	$=20.485$	$-q^1-q^3+q^5+q^7$	$=-2.613$	$0=0.000$	0.654

Table 114: Invariants for $S^3/P_{120} \times \mathbf{Z}_3$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$1-q^2+q^3$	$=0.382$	0	$=0.000$	0	$=0.000$	0.053
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$8+5q^2-2q^3+2q^4-5q^5$	$=13.345$	0	$=0.000$	0	$=0.000$	0.718
8	$11+6q^2-6q^6$	$=19.485$	0	$=0.000$	0	$=0.000$	0.713

Table 115: Invariants for $S^3/Q_{32} \times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	$-4q^1+4q^3$	$=-5.657$	2	$=2.000$	0.086
5	$3+q^2-q^3$	$=3.618$	$-6-2q^2+2q^3$	$=-7.236$	$3+q^2-q^3$	$=3.618$	0.000
6	10	$=10.000$	$8q^1-4q^3$	$=6.928$	6	$=6.000$	1.911
7	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
8	8	$=8.000$	0	$=0.000$	$4-2q^2+2q^6$	$=1.172$	0.336

Table 116: Invariants for $S^3/Q_{20} \times \mathbf{Z}_7$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$1-q^2+q^3$	$=0.382$	$0=0.000$	$1-q^2+q^3$	$=0.382$	0.106
6	4	$=4.000$	$0=0.000$	0	$=0.000$	0.333
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$0=0.000$	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	1.000
8	$12+6q^2-6q^6$	$=20.485$	$0=0.000$	$8+4q^2-4q^6$	$=13.657$	1.250

Table 117: Invariants for S^3/D_{112}:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	1 =1.000	1.000
4	2 =2.000	0 =0.000	2 =2.000	1.000
5	$4+q^2-q^3$ =4.618	0 =0.000	$4+q^2-q^3$ =4.618	1.276
6	4 =4.000	0 =0.000	0 =0.000	0.333
7	$2-q^2-q^3+q^4+q^5$ =0.308	0 =0.000	$2-q^2-q^3+q^4+q^5$ =0.308	0.033
8	$8+4q^2-4q^6$ =13.657	0 =0.000	$8+4q^2-4q^6$ =13.657	1.000

Table 118: Invariants for $S^3/Q_{16}\times \mathbf{Z}_7$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	-2 =−2.000	1 =1.000	0.000
4	4 =4.000	0 =0.000	2 =2.000	1.500
5	$2+q^2-q^3$ =2.618	$-4-2q^2+2q^3$ =−5.236	$2+q^2-q^3$ =2.618	0.000
6	10 =10.000	$-8q^1+4q^3$ =−6.928	6 =6.000	0.756
7	$6+3q^2-q^3+q^4-3q^5$ =9.296	$-12-6q^2+2q^3-2q^4+6q^5$ = −18.592	$6+3q^2-q^3+q^4-3q^5$ =9.296	0.000
8	$16+8q^2-8q^6$ =27.314	$-4q^1-4q^3+4q^5+4q^7$ = −10.453	$12+6q^2-6q^6$ =20.485	1.367

Table 119: Invariants for $S^3/Q_{16}\times \mathbf{Z}_5$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	-2 =−2.000	1 =1.000	0.000
4	4 =4.000	0 =0.000	2 =2.000	1.500
5	$3+q^2-q^3$ =3.618	$-6-2q^2+2q^3$ =−7.236	$3+q^2-q^3$ =3.618	0.000
6	10 =10.000	$-8q^1+4q^3$ =−6.928	6 =6.000	0.756
7	$8+5q^2-2q^3+2q^4-5q^5$ =13.345	$-16-10q^2+4q^3-4q^4+10q^5$ = −26.690	$8+5q^2-2q^3+2q^4-5q^5$ =13.345	0.000
8	$16+8q^2-8q^6$ =27.314	$-12q^1-4q^3+4q^5+12q^7$ = −25.235	$12+6q^2-6q^6$ =20.485	0.826

Table 120: Invariants for $S^3/Q_{12}\times \mathbf{Z}_7$:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	1 =1.000	1.000
4	2 =2.000	0 =0.000	0 =0.000	0.500
5	$5+2q^2-2q^3$ =6.236	0 =0.000	$5+2q^2-2q^3$ =6.236	1.724
6	10 =10.000	0 =0.000	6 =6.000	1.333
7	$6+3q^2-q^3+q^4-3q^5$ =9.296	0 =0.000	$6+3q^2-q^3+q^4-3q^5$ =9.296	1.000
8	$12+6q^2-6q^6$ =20.485	0 =0.000	$8+4q^2-4q^6$ =13.657	1.250

Table 121: Invariants for S^3/D_{160}:

r	$TV_0(M)_q$	$TV_1(M)_q$	$TV_2(M)_q$	TV^*
3	1 =1.000	0 =0.000	1 =1.000	1.000
4	2 =2.000	0 =0.000	2 =2.000	1.000
5	$1-q^2+q^3$ =0.382	0 =0.000	$1-q^2+q^3$ =0.382	0.106
6	4 =4.000	0 =0.000	0 =0.000	0.333
7	$7+2q^2-2q^5$ =9.494	0 =0.000	$7+2q^2-2q^5$ =9.494	1.021
8	$8+4q^2-4q^6$ =13.657	0 =0.000	$8+4q^2-4q^6$ =13.657	1.000

Table 122: Invariants for $S^3/Q_8 \times \mathbf{Z}_7$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$5+2q^2-2q^3$	=6.236	0 =0.000	$15+6q^2-6q^3$	=18.708	3.447
6	10	=10.000	0 =0.000	18	=18.000	2.333
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	0 =0.000	$18+9q^2-3q^3+3q^4-9q^5$	=27.888	2.000
8	$12+4q^2-4q^6$	=17.657	0 =0.000	$24+6q^2-6q^6$	=32.485	1.836

Table 123: Invariants for S^3/D_{80}:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$5+3q^2-3q^3$	=6.854	0 =0.000	$5+3q^2-3q^3$	=6.854	1.894
6	4	=4.000	0 =0.000	0	=0.000	0.333
7	$12+7q^2-2q^3+2q^4-7q^5$	=19.839	0 =0.000	$12+7q^2-2q^3+2q^4-7q^5$	=19.839	2.134
8	$8+4q^2-4q^6$	=13.657	0 =0.000	$8+4q^2-4q^6$	=13.657	1.000

Table 124: Invariants for $S^3/P_{120} \times \mathbf{Z}_{23}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$5+2q^2-2q^5$	=7.494	0	=0.000	0	=0.000	0.403
8	$9+4q^2-4q^6$	=14.657	0	=0.000	0	=0.000	0.537

Table 125: Invariants for $S^3/P_{24} \times \mathbf{Z}_7$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	0	=0.000	0	=0.000	0.500
8	$11+6q^2-6q^6$	=19.485	0	=0.000	0	=0.000	0.713

Table 126: Invariants for S^3/P'_{216}:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$4+2q^2-2q^5$	=6.494	0	=0.000	0	=0.000	0.349
8	$11+6q^2-6q^6$	=19.485	0	=0.000	0	=0.000	0.713

Table 127: Invariants for $S^3/P_{48} \times \mathbf{Z}_{11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	0	$=0.000$	0.146
5	$2+q^2-q^3$	$=2.618$	$-2-q^2+q^3$	$=-2.618$	0	$=0.000$	0.000
6	10	$=10.000$	0	$=0.000$	0	$=0.000$	0.833
7	$4+2q^2-2q^5$	$=6.494$	$-4-2q^2+2q^5$	$=-6.494$	0	$=0.000$	0.000
8	$12+6q^2-6q^6$	$=20.485$	$-3q^1-q^3+q^5+3q^7$	$=-6.309$	0	$=0.000$	0.519

Table 128: Invariants for $S^3/P_{24} \times \mathbf{Z}_{11}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$5+2q^2-2q^3$	$=6.236$	0	$=0.000$	0	$=0.000$	0.862
6	3	$=3.000$	0	$=0.000$	0	$=0.000$	0.250
7	2	$=2.000$	0	$=0.000$	0	$=0.000$	0.108
8	$9+4q^2-4q^6$	$=14.657$	0	$=0.000$	0	$=0.000$	0.537

Table 129: Invariants for $S^3/P_{120} \times \mathbf{Z}_{17}$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$1-q^2+q^3$	$=0.382$	0	$=0.000$	0	$=0.000$	0.053
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$5+2q^2-q^3+q^4-2q^5$	$=7.049$	0	$=0.000$	0	$=0.000$	0.379
8	$9+4q^2-4q^6$	$=14.657$	0	$=0.000$	0	$=0.000$	0.537

Table 130: Invariants for $ZF(1)$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$4+q^2-q^3$	$=4.618$	0	$=0.000$	0	$=0.000$	0.638
6	9	$=9.000$	0	$=0.000$	0	$=0.000$	0.750
7	$6+3q^2-2q^3+2q^4-3q^5$	$=8.851$	0	$=0.000$	0	$=0.000$	0.476
8	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.037

Table 131: Invariants for $ZF(2)$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	$=1.000$	-1	$=-1.000$	$0=0.000$	0.000
4	2	$=2.000$	$-q^1+q^3$	$=-1.414$	$0=0.000$	0.146
5	$5+2q^2-2q^3$	$=6.236$	$-5-2q^2+2q^3$	$=-6.236$	$0=0.000$	0.000
6	12	$=12.000$	$-12q^1+6q^3$	$=-10.392$	$0=0.000$	0.134
7	$9+6q^2-2q^3+2q^4-6q^5$	$=15.592$	$-9-6q^2+2q^3-2q^4+6q^5$	$=-15.592$	$0=0.000$	0.000
8	$20+10q^2-10q^6$	$=34.142$	$-9q^1-3q^3+3q^5+9q^7$	$=-18.926$	$0=0.000$	0.557

Table 132: Invariants for $ZF(3)$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	9	=9.000	0	=0.000	0	=0.000	0.750
7	$3+q^3-q^4$	=3.445	0	=0.000	0	=0.000	0.185
8	$3+2q^2-2q^6$	=5.828	0	=0.000	0	=0.000	0.213

Table 133: Invariants for $S^3/Q_{28}\times \mathbf{Z}_5$:

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$3+q^2-q^3$	=3.618	0	=0.000	$3+q^2-q^3$	=3.618	1.000
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$2-q^2+3q^3-3q^4+q^5$	=2.088	0	=0.000	$2-q^2+3q^3-3q^4+q^5$	=2.088	0.225
8	$4+2q^2-2q^6$	=6.828	0	=0.000	0	=0.000	0.250

Table 134: Invariants for S^3/D_{96}:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$5+2q^2-2q^3$	=6.236	0 =0.000	$5+2q^2-2q^3$	=6.236	1.724
6	10	=10.000	0 =0.000	6	=6.000	1.333
7	$5+2q^2-q^3+q^4-2q^5$	=7.049	0 =0.000	$5+2q^2-q^3+q^4-2q^5$	=7.049	0.758
8	$8+4q^2-4q^6$	=13.657	0 =0.000	$8+4q^2-4q^6$	=13.657	1.000

Table 135: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}}$:

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	4	=4.000	2.000
5	$4+q^2-q^3$	=4.618	0 =0.000	$12+3q^2-3q^3$	=13.854	2.553
6	22	=22.000	0 =0.000	30	=30.000	4.333
7	$16+10q^2-4q^3+4q^4-10q^5$	=26.690	0 =0.000	$48+30q^2-12q^3+12q^4-30q^5$	=80.069	5.742
8	$32+16q^2-16q^6$	=54.627	0 =0.000	$64+32q^2-32q^6$	=109.255	6.000

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Table 136: Invariants for $M(S^3,(2,1),(3,1),(7,-6))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	0	=0.000	0	=0.000	0	=0.000	0.000

Table 137: Invariants for $M(S^3,(2,1),(4,1),(5,-4))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0	=0.000	0.854
5	$2+q^2-q^3$	=2.618	$-2-q^2+q^3$	=−2.618	0	=0.000	0.000
6	4	=4.000	$4q-2q^3$	=3.464	0	=0.000	0.622
7	$5+2q^2-2q^5$	=7.494	$-5-2q^2+2q^5$	=−7.494	0	=0.000	0.000

Table 138: Invariants for $M(S^3,(2,1),(2,1),(7,-6))$, S^3/Q_{28}

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	0	=0.000	0.500
5	$4+q^2-q^3$	=4.618	0 =0.000	$4+q^2-q^3$	=4.618	1.276
6	4	=4.000	0 =0.000	0	=0.000	0.333
7	$14+11q^2-5q^3+5q^4-11q^5$	=25.492	0 =0.000	$14+11q^2-5q^3+5q^4-11q^5$	=25.492	2.742

Table 139: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}-1 & 0\\ -2 & 1\end{pmatrix}}$, $M(S^2,(2,1),(2,1),(2,1),(2,1))$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	0	=0.000	0 =0.000	0	=0.000	0.000
4	8	=8.000	0 =0.000	28	=28.000	9.000
5	$11+4q^2-4q^3$	=13.472	0 =0.000	$77+28q^2-28q^3$	=94.305	14.894
6	46	=46.000	0 =0.000	198	=198.000	20.333
7	$40+18q^2-6q^3+6q^4-18q^5$	=59.775	0 =0.000	$280+126q^2-42q^3+42q^4-126q^5$	=418.428	25.721

Table 140: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}2 & -1\\ 1 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	4	=4.000	2.000
5	$3-q^2+q^3$	=2.382	0 =0.000	$9-3q^2+3q^3$	=7.146	1.317
6	16	=16.000	0 =0.000	-12	=−12.000	0.333
7	$4-4q^2+4q^3-4q^4+4q^5$	=0.792	0 =0.000	$12-12q^2+12q^3-12q^4+12q^5$	=2.377	0.170

Table 141: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-4	$=-4.000$	3	$=3.000$	0.000
4	8	$=8.000$	$-8q+8q^3$	$=-11.314$	12	$=12.000$	2.172
5	$7+4q^2-4q^3$	$=9.472$	$-28-16q^2+16q^3$	$=-37.889$	$21+12q^2-12q^3$	$=28.416$	0.000
6	30	$=30.000$	$-48q+24q^3$	$=-41.569$	54	$=54.000$	3.536
7	$12+6q^2-2q^3+2q^4-6q^5$	$=18.592$	$-48-24q^2+8q^3-8q^4+24q^5$	$=-74.367$	$36+18q^2-6q^3+6q^4-18q^5$	$=55.775$	0.000

Table 142: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	0	$=0.000$	0	$=0.000$	1.000
5	$1-q^2+q^3$	$=0.382$	0	$=0.000$	$3-3q^2+3q^3$	$=1.146$	0.211
6	16	$=16.000$	0	$=0.000$	-12	$=-12.000$	0.333
7	4	$=4.000$	0	$=0.000$	12	$=12.000$	0.861

Table 143: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}-2 & 1\\ -1 & 0\end{pmatrix}}$, $M(RP^2,(2,1),(2,3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	0	$=0.000$	12	$=12.000$	4.000
5	$7+2q^2-2q^3$	$=8.236$	0	$=0.000$	$21+6q^2-6q^3$	$=24.708$	4.553
6	22	$=22.000$	0	$=0.000$	30	$=30.000$	4.333
7	$12+2q^2-2q^5$	$=14.494$	0	$=0.000$	$36+6q^2-6q^5$	$=43.482$	3.118

Table 144: Invariants for $\frac{K\tilde{\times}I\cup K\tilde{\times}I}{\begin{pmatrix}1 & -2\\ 1 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-4	$=-4.000$	3	$=3.000$	0.000
4	8	$=8.000$	$-8q^1+8q^3$	$=-11.314$	8	$=8.000$	1.172
5	$5+3q^2-3q^3$	$=6.854$	$-20-12q^2+12q^3$	$=-27.416$	$15+9q^2-9q^3$	$=20.562$	0.000
6	28	$=28.000$	$-32q^1+16q^3$	$=-27.713$	24	$=24.000$	2.024
7	$8+4q^2-4q^5$	$=12.988$	$-32-16q^2+16q^5$	$=-51.952$	$24+12q^2-12q^5$	$=38.964$	0.000

Table 145: Invariants for $\frac{(D^2,(2,1),(3,-1))\cup K\tilde{\times}I}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}$, $M(S^2,(2,1),(2,-1),(2,1),(3,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$10+3q^2-3q^3$	=11.854	0 =0.000	$30+9q^2-9q^3$	=35.562	6.553
6	28	=28.000	0 =0.000	72	=72.000	8.333
7	$29+14q^2-5q^3+5q^4-14q^5$	=44.233	0 =0.000	$87+42q^2-15q^3+15q^4-42q^5$	=132.698	9.517

Table 146: Invariants for $\frac{(D^2,(2,1),(3,-1))\cup K\tilde{\times}I}{\begin{pmatrix}-1&1\\-1&0\end{pmatrix}}$, $M(RP^2,(2,1),(3,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$7+2q^2-2q^3$	=8.236	0	=0.000	$7+2q^2-2q^3$	=8.236	2.276
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	4	=4.000	0	=0.000	4	=4.000	0.430

Table 147: Invariants for $\frac{(D^2,(2,1),(3,-2))\cup K\tilde{\times}I}{\begin{pmatrix}-1&1\\-1&0\end{pmatrix}}$, $M(RP^2,(2,1),(3,2))$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$8+4q^2-4q^3$	=10.472	0 =0.000	$8+4q^2-4q^3$	=10.472	2.894
6	6	=6.000	0 =0.000	6	=6.000	1.000
7	$16+11q^2-4q^3+4q^4-11q^5$	=27.937	0 =0.000	$16+11q^2-4q^3+4q^4-11q^5$	=27.937	3.005

Table 148: Invariants for $\frac{(D^2,(2,1),(3,-1))\cup K\tilde{\times}I}{\begin{pmatrix}0&-1\\1&-1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	0	=0.000	0.500
5	$1-q^2+q^3$	=0.382	0 =0.000	$1-q^2+q^3$	=0.382	0.106
6	4	=4.000	0 =0.000	0	=0.000	0.333
7	$3-2q^2+3q^3-3q^4+2q^5$	=1.841	0 =0.000	$3-2q^2+3q^3-3q^4+2q^5$	=1.841	0.198

Table 149: Invariants for $\dfrac{(D^2,(2,1),(3,-2))\cup K\tilde{\times}I}{\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$2-q^2+q^3$	=1.382	0	=0.000	$2-q^2+q^3$	=1.382	0.382
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$5+2q^3-2q^4$	=5.890	0	=0.000	$5+2q^3-2q^4$	=5.890	0.634

Table 150: Invariants for $\dfrac{(D^2,(2,1),(3,-1))\cup M\tilde{\times}S^1}{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$4+q^2-q^3$	=4.618	0 =0.000	$4+q^2-q^3$	=4.618	1.276
6	6	=6.000	0 =0.000	6	=6.000	1.000
7	$9+2q^2+q^3-q^4-2q^5$	=11.939	0 =0.000	$9+2q^2+q^3-q^4-2q^5$	=11.939	1.284

Table 151: Invariants for $\dfrac{(D^2,(2,1),(3,-1))\cup M\tilde{\times}S^1}{\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$11+4q^2-4q^3$	=13.472	0 =0.000	$33+12q^2-12q^3$	=40.416	7.447
6	28	=28.000	0 =0.000	72	=72.000	8.333
7	$24+12q^2-4q^3+4q^4-12q^5$	=37.184	0 =0.000	$72+36q^2-12q^3+12q^4-36q^5$	=111.551	8.000

Table 152: Invariants for $\dfrac{(D^2,(2,1),(3,-2))\cup M\tilde{\times}S^1}{\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$11+4q^2-4q^3$	=13.472	0 =0.000	$33+12q^2-12q^3$	=40.416	7.447
6	28	=28.000	0 =0.000	72	=72.000	8.333
7	$128-213q^2+403q^3-403q^4+213q^5$	=41.745	0 =0.000	$1140-2327q^2+4251q^3-4251q^4+2327q^5$	=130.151	9.246

Table 153: Invariants for $\frac{(D^2,(2,1),(3,-1))\cup M\widetilde{\times}S^1}{\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$3-q^2+q^3$	$=2.382$	$0=0.000$	$3-q^2+q^3$	$=2.382$	0.658
6	4	$=4.000$	$0=0.000$	0	$=0.000$	0.333
7	$3-2q^2+q^3-q^4+2q^5$	$=0.951$	$0=0.000$	$3-2q^2+q^3-q^4+2q^5$	$=0.951$	0.102

Table 154: Invariants for $\frac{(D^2,(2,1),(3,-2))\cup M\widetilde{\times}S^1}{\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$4+q^2-q^3$	$=4.618$	$0=0.000$	$4+q^2-q^3$	$=4.618$	1.276
6	4	$=4.000$	$0=0.000$	0	$=0.000$	0.333
7	$9+6q^2-2q^3+2q^4-6q^5$	$=15.592$	$0=0.000$	$9+6q^2-2q^3+2q^4-6q^5$	$=15.592$	1.677

Table 155: Invariants for $\frac{T^2\times I}{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}}$, $M(T^2,2)$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-4	$=-4.000$	3	$=3.000$	0.000
4	8	$=8.000$	$-8q+8q^3$	$=-11.314$	12	$=12.000$	2.172
5	$7+4q^2-4q^3$	$=9.472$	$-28-16q^2+16q^3$	$=-37.889$	$21+12q^2-12q^3$	$=28.416$	0.000
6	30	$=30.000$	$-48q+24q^3$	$=-41.569$	54	$=54.000$	3.536
7	$12+6q^2-2q^3+2q^4-6q^5$	$=18.592$	$-48-24q^2+8q^3-8q^4+24q^5$	$=-74.367$	$36+18q^2-6q^3+6q^4-18q^5$	$=55.775$	0.000

Table 156: Invariants for $\frac{T^2\times I}{\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	$-4q+4q^3$	$=-5.657$	0	$=0.000$	-0.414
5	$3+q^2-q^3$	$=3.618$	$-6-2q^2+2q^3$	$=-7.236$	$3+q^2-q^3$	$=3.618$	0.000
6	12	$=12.000$	0	$=0.000$	0	$=0.000$	1.000
7	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000

Table 157: Invariants for $\frac{T^2\times I}{\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}}$, $M(K,2)$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	$-4q+4q^3$	$=-5.657$	0	$=0.000$	-0.414
5	$3+q^2-q^3$	$=3.618$	$-6-2q^2+2q^3$	$=-7.236$	$3+q^2-q^3$	$=3.618$	0.000
6	12	$=12.000$	0	$=0.000$	0	$=0.000$	1.000
7	0	$=0.000$	0	$=0.000$	0	$=0.000$	0.000

Table 158: Invariants for $\frac{T^2\times I}{\begin{pmatrix}-2 & -1\\ -1 & -1\end{pmatrix}}$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	-2	$=-2.000$	0.000
5	$2-q^2+q^3$	$=1.382$	$0=0.000$	$2-q^2+q^3$	$=1.382$	0.382
6	6	$=6.000$	$0=0.000$	-6	$=-6.000$	0.000
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$0=0.000$	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	1.000

Table 159: Invariants for $M(S^3,(2,1),(2,1),(6,-1))$, $S^3/Q_{24}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	0	$=0.000$	-2	$=-2.000$	0.500
5	$3+q^2-q^3$	$=3.618$	0	$=0.000$	$9+3q^2-3q^3$	$=10.854$	2.000
6	16	$=16.000$	0	$=0.000$	12	$=12.000$	2.333
7	1	$=1.000$	0	$=0.000$	3	$=3.000$	0.215

Table 160: Invariants for $M(S^3,(2,1),(2,1),(9,-2))$, $S^3/Q_{36}\times Z_7$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$2+q^2-q^3$	$=2.618$	$0=0.000$	$2+q^2-q^3$	$=2.618$	0.724
6	10	$=10.000$	$0=0.000$	6	$=6.000$	1.333
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$0=0.000$	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	1.000

Table 161: Invariants for $M(S^3,(2,1),(2,1),(11,-3))$, S^3/D_{352}

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	$2+q^2-q^3$	$=2.618$	0.724
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	$5+2q^2-q^3+q^4-2q^5$	$=7.049$	0	$=0.000$	$5+2q^2-q^3+q^4-2q^5$	$=7.049$	0.758

Table 162: Invariants for $M(S^3,(2,1),(2,1),(10,-3))$, $S^3/Q_{40}\times Z_7$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	3	$=3.000$	2.000
4	4	$=4.000$	$0=0.000$	-2	$=-2.000$	0.500
5	$1-q^2+q^3$	$=0.382$	$0=0.000$	$3-3q^2+3q^3$	$=1.146$	0.211
6	10	$=10.000$	$0=0.000$	-6	$=-6.000$	0.333
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$0=0.000$	$18+9q^2-3q^3+3q^4-9q^5$	$=27.888$	2.000

Table 163: Invariants for $M(S^3,(2,1),(2,1),(11,-4))$, $S^3/Q_{44}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	0	$=0.000$	0.500
5	$2+q^2-q^3$	$=2.618$	0	$=0.000$	$2+q^2-q^3$	$=2.618$	0.724
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	0	$=0.000$	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	1.000

Table 164: Invariants for $M(S^3,(2,1),(2,1),(13,-5))$, S^3/D_{416}

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	$5+2q^2-2q^3$	$=6.236$	0	$=0.000$	$5+2q^2-2q^3$	$=6.236$	1.724
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	$2+q^2-q^5$	$=3.247$	0	$=0.000$	$2+q^2-q^5$	$=3.247$	0.349

Table 165: Invariants for $M(S^3,(2,1),(2,1),(12,-5))$, $S^3/Q_{48}\times Z_7$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	0	$=0.000$	2	$=2.000$	1.500
5	$5+2q^2-2q^3$	$=6.236$	$-10-4q^2+4q^3$	$=-12.472$	$5+2q^2-2q^3$	$=6.236$	0.000
6	16	$=16.000$	$-32q+16q^3$	$=-27.713$	12	$=12.000$	0.024
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	$-12-6q^2+2q^3-2q^4+6q^5$	$=-18.592$	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	0.000

Table 166: Invariants for $M(S^3,(2,1),(2,1),(9,-4))$, $S^3/Q_{36}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	$=1.000$	$0=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	$0=0.000$	0	$=0.000$	0.500
5	$3+q^2-q^3$	$=3.618$	$0=0.000$	$3+q^2-q^3$	$=3.618$	1.000
6	10	$=10.000$	$0=0.000$	6	$=6.000$	1.333
7	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	$0=0.000$	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	1.845

Table 167: Invariants for $M(S^3,(2,1),(2,1),(9,-5))$, S^3/D_{144}

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	1	=1.000	0 =0.000	1	=1.000	0.276
6	10	=10.000	0 =0.000	6	=6.000	1.333
7	$12+7q^2-2q^3+2q^4-7q^5$	=19.839	0 =0.000	$12+7q^2-2q^3+2q^4-7q^5$	=19.839	2.134

Table 168: Invariants for $M(S^3,(2,1),(2,1),(12,-7))$, $S^3/Q_{48}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	=-2.000	1	=1.000	0.000
4	4	=4.000	0	=0.000	2	=2.000	1.500
5	$3+q^2-q^3$	=3.618	$-6-2q^2+2q^3$	=-7.236	$3+q^2-q^3$	=3.618	0.000
6	16	=16.000	$-32q+16q^3$	=-27.713	12	=12.000	0.024
7	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	$-22-12q^2+6q^3-6q^4+12q^5$	=-34.294	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	0.000

Table 169: Invariants for $M(S^3,(2,1),(2,1),(13,-8))$, $S^3/Q_{52}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$3+q^2-q^3$	=3.618	0	=0.000	$3+q^2-q^3$	=3.618	1.000
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	1	=1.000	0	=0.000	1	=1.000	0.108

Table 170: Invariants for $M(S^3,(2,1),(2,1),(11,-7))$, S^3/D_{176}

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	1	=1.000	0	=0.000	1	=1.000	0.276
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$5+2q^2-2q^5$	=7.494	0	=0.000	$5+2q^2-2q^5$	=7.494	0.806

Table 171: Invariants for $M(S^3,(2,1),(2,1),(10,-7))$, $S^3/Q_{40}\times Z_3$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	3	=3.000	2.000
4	4	=4.000	0	=0.000	-2	=-2.000	0.500
5	$1-q^2+q^3$	=0.382	0	=0.000	$3-3q^2+3q^3$	=1.146	0.211
6	12	=12.000	0	=0.000	-12	=-12.000	0.000
7	$5+2q^2-2q^5$	=7.494	0	=0.000	$15+6q^2-6q^5$	=22.482	1.612

Table 172: Invariants for $M(S^3,(2,1),(2,1),(11,-8))$, $S^3/Q_{44}\times Z_3$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$2+q^2-q^3$	=2.618	0	=0.000	$2+q^2-q^3$	=2.618	0.724
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$5+2q^2-2q^5$	=7.494	0	=0.000	$5+2q^2-2q^5$	=7.494	0.806

Table 173: Invariants for $M(S^3,(2,1),(2,1),(9,-7))$, S^3/D_{72}

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$2+q^2-q^3$	=2.618	0	=0.000	$2+q^2-q^3$	=2.618	0.724
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$5+2q^2-2q^5$	=7.494	0	=0.000	$5+2q^2-2q^5$	=7.494	0.806

Table 174: Invariants for $S^3/Q_8\times Z_9$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	3	=3.000	2.000
4	4	=4.000	0 =0.000	6	=6.000	2.500
5	$4+q^2-q^3$	=4.618	0 =0.000	$12+3q^2-3q^3$	=13.854	2.553
6	12	=12.000	0 =0.000	36	=36.000	4.000
7	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	0 =0.000	$33+18q^2-9q^3+9q^4-18q^5$	=51.440	3.689

Table 175: Invariants for $M(S^3,(2,1),(3,2),(5,-1))$, $S^3/P_{120}\times Z_{29}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$6-2q^2+6q^3-6q^4+2q^5$	=6.176	0	=0.000	0	=0.000	0.332

Table 176: Invariants for $M(S^3,(2,1),(3,2),(7,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	3	=3.000	0	=0.000	0	=0.000	0.415
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$-2q^2+8q^3-8q^4+2q^5$	=1.066	0	=0.000	0	=0.000	0.057

Table 177: *Invariants for* $M(S^3,(2,1),(3,2),(8,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	3	=3.000	0	=0.000	0	=0.000	0.415
6	5	=5.000	0	=0.000	0	=0.000	0.417
7	$1-2q^2+7q^3-7q^4+2q^5$	=1.621	0	=0.000	0	=0.000	0.087

Table 178: *Invariants for* $M(S^3,(2,1),(3,2),(7,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	2	=2.000	0	=0.000	0	=0.000	0.276
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$4-3q^2+6q^3-6q^4+3q^5$	=2.929	0	=0.000	0	=0.000	0.158

Table 179: *Invariants for* $M(S^3,(2,1),(3,2),(7,-4))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$4-3q^2+6q^3-6q^4+3q^5$	=2.929	0	=0.000	0	=0.000	0.158

Table 180: *Invariants for* $M(S^3,(2,1),(3,2),(8,-5))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	4	=4.000	0	=0.000	0	=0.000	0.553
6	7	=7.000	0	=0.000	0	=0.000	0.583
7	$8q^3-8q^4$	=3.560	0	=0.000	0	=0.000	0.192

Table 181: *Invariants for* $M(S^3,(2,1),(3,2),(7,-5))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	4	=4.000	0	=0.000	0	=0.000	0.553
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$-2q^2+8q^3-8q^4+2q^5$	=1.066	0	=0.000	0	=0.000	0.057

Table 182: *Invariants for* $M(S^3,(2,1),(3,2),(5,-4))$, $S^3/P_{120}\times Z_{11}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$6-3q^2+7q^3-7q^4+3q^5$	=5.374	0	=0.000	0	=0.000	0.289

Table 183: Invariants for $S^3/Q_{12} \times Z_{11}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$4+q^2-q^3$	=4.618	0	=0.000	$4+q^2-q^3$	=4.618	1.276
6	10	=10.000	0	=0.000	6	=6.000	1.333
7	$5+2q^2-2q^5$	=7.494	0	=0.000	$5+2q^2-2q^5$	=7.494	0.806

Table 184: Invariants for $M(S^3,(2,1),(3,1),(5,-1))$, $S^3/P_{120} \times Z_{19}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+3q^2-3q^3$	=6.854	0	=0.000	0	=0.000	0.947
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$5+2q^2-2q^5$	=7.494	0	=0.000	0	=0.000	0.403

Table 185: Invariants for $M(S^3,(2,1),(3,1),(7,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$1-q^2+2q^3-2q^4+q^5$	=0.643	0	=0.000	0	=0.000	0.035

Table 186: Invariants for $M(S^3,(2,1),(3,1),(8,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	$=-1.000$	0	=0.000	0.000
4	2	=2.000	$-q+q^3$	$=-1.414$	0	=0.000	0.146
5	$5+2q^2-2q^3$	=6.236	$-5-2q^2+2q^3$	$=-6.236$	0	=0.000	0.000
6	10	=10.000	0	=0.000	0	=0.000	0.833
7	0	=0.000	0	=0.000	0	=0.000	0.000

Table 187: Invariants for $M(S^3,(2,1),(3,1),(7,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$2-q^3+q^4$	=1.555	0	=0.000	0	=0.000	0.084

Table 188: Invariants for $M(S^3,(2,1),(3,1),(7,-4))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$2-q^3+q^4$	=1.555	0	=0.000	0	=0.000	0.084

Table 189: Invariants for $M(S^3,(2,1),(3,1),(8,-5))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q+q^3$	$=-1.414$	0	$=0.000$	0.146
5	$3+q^2-q^3$	$=3.618$	$-3-q^2+q^3$	$=-3.618$	0	$=0.000$	0.000
6	10	$=10.000$	0	$=0.000$	0	$=0.000$	0.833
7	$2+q^2-q^5$	$=3.247$	$-2-q^2+q^5$	$=-3.247$	0	$=0.000$	0.000

Table 190: Invariants for $M(S^3,(2,1),(3,1),(7,-5))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$3+q^2-q^3$	$=3.618$	0	$=0.000$	0	$=0.000$	0.500
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$1-q^2+2q^3-2q^4+q^5$	$=0.643$	0	$=0.000$	0	$=0.000$	0.035

Table 191: Invariants for $S^3/D_{24}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.553
6	4	$=4.000$	0	$=0.000$	0	$=0.000$	0.333
7	$5-2q^2+8q^3-8q^4+2q^5$	$=6.066$	0	$=0.000$	0	$=0.000$	0.326

Table 192: Invariants for $M(S^3,(2,1),(2,1),(5,4))$, $S^3/Q_{20}\times Z_9$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	0	$=0.000$	0.500
5	$5+3q^2-3q^3$	$=6.854$	0	$=0.000$	$5+3q^2-3q^3$	$=6.854$	1.894
6	6	$=6.000$	0	$=0.000$	6	$=6.000$	1.000
7	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	0	$=0.000$	$11+6q^2-3q^3+3q^4-6q^5$	$=17.147$	1.845

Table 193: Invariants for $M(S^3,(2,1),(2,1),(7,5))$, $S^3/D_{112}\times Z_3$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	1	$=1.000$	1.000
4	2	$=2.000$	0	$=0.000$	2	$=2.000$	1.000
5	$5+2q^2-2q^3$	$=6.236$	0	$=0.000$	$5+2q^2-2q^3$	$=6.236$	1.724
6	6	$=6.000$	0	$=0.000$	6	$=6.000$	1.000
7	$2-q^2+3q^3-3q^4+q^5$	$=2.088$	0	$=0.000$	$2-q^2+3q^3-3q^4+q^5$	$=2.088$	0.225

Table 194: Invariants for $M(S^3,(2,1),(2,1),(8,5))$, $S^3/Q_{32}\times Z_{13}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	$-4q+4q^3$	$=-5.657$	2	$=2.000$	0.086
5	$5+2q^2-2q^3$	$=6.236$	$-10-4q^2+4q^3$	$=-12.472$	$5+2q^2-2q^3$	$=6.236$	0.000
6	10	$=10.000$	$-8q+4q^3$	$=-6.928$	6	$=6.000$	0.756
7	$2+q^2-q^5$	$=3.247$	$-4-2q^2+2q^5$	$=-6.494$	$2+q^2-q^5$	$=3.247$	0.000

Table 195: Invariants for $M(S^3,(2,1),(2,1),(7,4))$, $S^3/Q_{28}\times Z_{11}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$4+q^2-q^3$	=4.618	0	=0.000	$4+q^2-q^3$	=4.618	1.276
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$2-q^2-q^3+q^4+q^5$	=0.308	0	=0.000	$2-q^2-q^3+q^4+q^5$	=0.308	0.033

Table 196: Invariants for $M(S^3,(2,1),(2,1),(7,3))$, $S^3/D_{56}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$3+q^2-q^3$	=3.618	0	=0.000	$3+q^2-q^3$	=3.618	1.000
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$2-q^2-q^3+q^4+q^5$	=0.308	0	=0.000	$2-q^2-q^3+q^4+q^5$	=0.308	0.033

Table 197: Invariants for $M(S^3,(2,1),(2,1),(8,3))$, $S^3/Q_{32}\times Z_{11}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	=−2.000	1	=1.000	0.000
4	4	=4.000	$-4q+4q^3$	=−5.657	2	=2.000	0.086
5	$4+q^2-q^3$	=4.618	$-8-2q^2+2q^3$	=−9.236	$4+q^2-q^3$	=4.618	0.000
6	10	=10.000	$-8q+4q^3$	=−6.928	6	=6.000	0.756
7	$3+2q^2-q^3+q^4-2q^5$	=5.049	$-6-4q^2+2q^3-2q^4+4q^5$	=−10.098	$3+2q^2-q^3+q^4-2q^5$	=5.049	0.000

Table 198: Invariants for $M(S^3,(2,1),(2,1),(7,2))$, $S^3/Q_{28}\times Z_9$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$4+q^2-q^3$	=4.618	0	=0.000	$4+q^2-q^3$	=4.618	1.276
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$2-q^2+3q^3-3q^4+q^5$	=2.088	0	=0.000	$2-q^2+3q^3-3q^4+q^5$	=2.088	0.225

Table 199: Invariants for $M(S^3,(2,1),(2,1),(5,1))$, $S^3/D_{40}\times Z_3$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	2	=2.000	1.000
5	$5+3q^2-3q^3$	=6.854	0	=0.000	$5+3q^2-3q^3$	=6.854	1.894
6	6	=6.000	0	=0.000	6	=6.000	1.000
7	$7+2q^2-2q^5$	=9.494	0	=0.000	$7+2q^2-2q^5$	=9.494	1.021

Table 200: Invariants for $M(S^3,(2,1),(4,3),(4,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	0	$=0.000$	0	$=0.000$	1.000
5	$2+q^2-q^3$	$=2.618$	$-4-2q^2+2q^3$	$=-5.236$	$2+q^2-q^3$	$=2.618$	0.000
6	10	$=10.000$	$-8q+4q^3$	$=-6.928$	6	$=6.000$	0.756
7	$6+4q^2-2q^3+2q^4-4q^5$	$=10.098$	$-12-8q^2+4q^3-4q^4+8q^5$	$=-20.196$	$6+4q^2-2q^3+2q^4-4q^5$	$=10.098$	0.000

Table 201: Invariants for $M(S^3,(2,1),(4,3),(5,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	$=1.000$	-1	$=-1.000$	0 $=0.000$	0.000
4	2	$=2.000$	$q-q^3$	$=1.414$	0 $=0.000$	0.854
5	1	$=1.000$	-1	$=-1.000$	0 $=0.000$	0.000
6	4	$=4.000$	$-4q+2q^3$	$=-3.464$	0 $=0.000$	0.045
7	$8+5q^2-2q^3+2q^4-5q^5$	$=13.345$	$-8-5q^2+2q^3-2q^4+5q^5$	$=-13.345$	0 $=0.000$	0.000

Table 202: Invariants for $M(S^3,(2,1),(4,3),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
4	2	$=2.000$	$-q+q^3$	$=-1.414$	0	$=0.000$	0.146
5	1	$=1.000$	-1	$=-1.000$	0	$=0.000$	0.000
6	4	$=4.000$	$4q-2q^3$	$=3.464$	0	$=0.000$	0.622
7	$5+2q^2-2q^5$	$=7.494$	$-5-2q^2+2q^5$	$=-7.494$	0	$=0.000$	0.000

Table 203: Invariants for $M(S^3,(2,1),(4,3),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
4	4	$=4.000$	0	$=0.000$	0	$=0.000$	1.000
5	1	$=1.000$	-2	$=-2.000$	1	$=1.000$	0.000
6	10	$=10.000$	0	$=0.000$	-6	$=-6.000$	0.333
7	2	$=2.000$	-4	$=-4.000$	2	$=2.000$	0.000

Table 204: Invariants for $M(S^3,(2,1),(5,3),(5,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.500
4	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.250
5	$2-q^2+q^3$	$=1.382$	0	$=0.000$	0	$=0.000$	0.191
6	1	$=1.000$	0	$=0.000$	0	$=0.000$	0.083
7	$6+3q^2-q^3+q^4-3q^5$	$=9.296$	0	$=0.000$	0	$=0.000$	0.500

Table 205: Invariants for $M(S^3,(2,1),(5,3),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2-q^2+q^3$	=1.382	0	=0.000	0	=0.000	0.191
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$7+2q^2-2q^5$	=9.494	0	=0.000	0	=0.000	0.511

Table 206: Invariants for $M(S^3,(2,1),(5,3),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	$=-1.000$	0	=0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0	=0.000	0.854
5	1	=1.000	-1	$=-1.000$	0	=0.000	0.000
6	4	=4.000	$-4q+2q^3$	$=-3.464$	0	=0.000	0.045
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	$-6-3q^2+q^3-q^4+3q^5$	$=-9.296$	0	=0.000	0.000

Table 207: Invariants for $M(S^3,(2,1),(5,2),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2-q^2+q^3$	=1.382	0	=0.000	0	=0.000	0.191
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	0	=0.000	0	=0.000	0.922

Table 208: Invariants for $M(S^3,(2,1),(5,2),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	$=-1.000$	0	=0.000	0.000
4	2	=2.000	$-q+q^3$	$=-1.414$	0	=0.000	0.146
5	1	=1.000	-1	$=-1.000$	0	=0.000	0.000
6	6	=6.000	0	=0.000	0	=0.000	0.500
7	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	$-8-5q^2+2q^3-2q^4+5q^5$	$=-13.345$	0	=0.000	0.000

Table 209: Invariants for $M(S^3,(2,1),(2,1),(4,7))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	$=-2.000$	1	=1.000	0.000
4	4	=4.000	0	=0.000	2	=2.000	1.500
5	1	=1.000	-2	$=-2.000$	1	=1.000	0.000
6	10	=10.000	$8q-4q^3$	=6.928	6	=6.000	1.911
7	$5+2q^2-2q^5$	=7.494	$-10-4q^2+4q^5$	$=-14.988$	$5+2q^2-2q^5$	=7.494	0.000

Table 210: Invariants for $M(S^3,(2,1),(2,1),(5,8))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$1-q^2+q^3$	=0.382	0	=0.000	$1-q^2+q^3$	=0.382	0.106
6	4	=4.000	0	=0.000	0	=0.000	0.333
7	$7+2q^2-2q^5$	=9.494	0	=0.000	$7+2q^2-2q^5$	=9.494	1.021

Table 211: Invariants for $M(S^3,(2,1),(2,1),(5,7))$

r	$TV_0(M)_q$		$TV_1(M)_q$	$TV_2(M)_q$		TV^*
3	1	=1.000	0 =0.000	1	=1.000	1.000
4	2	=2.000	0 =0.000	2	=2.000	1.000
5	$1-q^2+q^3$	=0.382	0 =0.000	$1-q^2+q^3$	=0.382	0.106
6	6	=6.000	0 =0.000	6	=6.000	1.000
7	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	0 =0.000	$11+6q^2-3q^3+3q^4-6q^5$	=17.147	1.845

Table 212: Invariants for $M(S^3,(2,1),(2,1),(4,5))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-2	$=-2.000$	1	=1.000	0.000
4	4	=4.000	0	=0.000	2	=2.000	1.500
5	1	=1.000	-2	$=-2.000$	1	=1.000	0.000
6	12	=12.000	0	=0.000	12	=12.000	2.000
7	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	$-16-10q^2+4q^3-4q^4+10q^5$	$=-26.690$	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	0.000

Table 213: Invariants for $M(S^3,(3,2),(3,2),(4,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$9+6q^2-2q^3+2q^4-6q^5$	=15.592	0	=0.000	0	=0.000	0.839

Table 214: Invariants for $M(S^3,(3,2),(3,2),(5,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	=1.000	-1	$=-1.000$	0 =0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0 =0.000	0.854
5	$1-q^2+q^3$	=0.382	$-1+q^2-q^3$	$=-0.382$	0 =0.000	0.000
6	6	=6.000	0	=0.000	0 =0.000	0.500
7	$6+3q^2-q^3+q^4-3q^5$	=9.296	$-6-3q^2+q^3-q^4+3q^5$	$=-9.296$	0 =0.000	0.000

Table 215: Invariants for $M(S^3,(3,2),(3,2),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	2	=2.000	0	=0.000	0	=0.000	0.108

Table 216: Invariants for $M(S^3,(3,2),(3,2),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	0	=0.000	0	=0.000	0	=0.000	0.000

Table 217: Invariants for $M(S^3,(3,2),(3,1),(4,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$3+q^3-q^4$	=3.445	0	=0.000	0	=0.000	0.185

Table 218: Invariants for $M(S^3,(3,2),(3,1),(5,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$4+2q^2-2q^5$	=6.494	0	=0.000	0	=0.000	0.349

Table 219: Invariants for $M(S^3,(3,2),(3,1),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$-q+q^3$	=−1.414	0	=0.000	0.146
5	$1-q^2+q^3$	=0.382	$-1+q^2-q^3$	=−0.382	0	=0.000	0.000
6	6	=6.000	0	=0.000	0	=0.000	0.500
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	$-6-4q^2+2q^3-2q^4+4q^5$	=−10.098	0	=0.000	0.000

Table 220: Invariants for $M(S^3,(3,2),(3,1),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	1	=1.000	0	=0.000	0	=0.000	0.138
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+3q^2-2q^3+2q^4-3q^5$	=8.851	0	=0.000	0	=0.000	0.476

Table 221: Invariants for $M(S^3,(3,1),(3,1),(4,-1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$3+q^3-q^4$	=3.445	0	=0.000	0	=0.000	0.185

Table 222: Invariants for $M(S^3,(3,1),(3,1),(5,-2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$1-q^2+q^3$	=0.382	0	=0.000	$1-q^2+q^3$	=0.382	0.106
6	6	=6.000	0	=0.000	-6	$=-6.000$	0.000
7	2	=2.000	0	=0.000	2	=2.000	0.215

Table 223: Invariants for $M(S^3,(3,1),(3,1),(5,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	0	=0.000	0	=0.000	0.543

Table 224: Invariants for $M(S^3,(3,1),(3,1),(4,-3))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$2+q^2-q^3$	=2.618	0	=0.000	0	=0.000	0.362
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$9+6q^2-2q^3+2q^4-6q^5$	=15.592	0	=0.000	0	=0.000	0.839

Table 225: Invariants for $M(S^3,(2,1),(3,2),(4,3))$, $S^3/P_{48}\times Z_{23}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$	TV^*
3	1	=1.000	-1	$=-1.000$	0 =0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0 =0.000	0.854
5	1	=1.000	-1	$=-1.000$	0 =0.000	0.000
6	10	=10.000	0	=0.000	0 =0.000	0.833
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	$-6-4q^2+2q^3-2q^4+4q^5$	$=-10.098$	0 =0.000	0.000

Table 226: Invariants for $M(S^3,(2,1),(3,2),(5,3))$, $S^3/P_{120}\times Z_{53}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$5+2q^2-q^3+q^4-2q^5$	=7.049	0	=0.000	0	=0.000	0.379

Table 227: Invariants for $M(S^3,(2,1),(3,2),(5,2))$, $S^3/P_{120}\times Z_{47}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$5+2q^2-2q^5$	=7.494	0	=0.000	0	=0.000	0.403

Table 228: Invariants for $M(S^3,(2,1),(3,2),(4,1))$, $S^3/P_{48}\times Z_{17}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0	=0.000	0.854
5	1	=1.000	-1	=−1.000	0	=0.000	0.000
6	10	=10.000	0	=0.000	0	=0.000	0.833
7	$4+2q^2-2q^5$	=6.494	$-4-2q^2+2q^5$	=−6.494	0	=0.000	0.000

Table 229: Invariants for $M(S^3,(2,1),(3,1),(4,3))$, $S^3/P_{48}\times Z_{19}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$-q+q^3$	=−1.414	0	=0.000	0.146
5	$2+q^2-q^3$	=2.618	$-2-q^2+q^3$	=−2.618	0	=0.000	0.000
6	10	=10.000	0	=0.000	0	=0.000	0.833
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	$-6-4q^2+2q^3-2q^4+4q^5$	=−10.098	0	=0.000	0.000

Table 230: Invariants for $M(S^3,(2,1),(3,1),(5,3))$, $S^3/P_{120}\times Z_{43}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$8+5q^2-2q^3+2q^4-5q^5$	=13.345	0	=0.000	0	=0.000	0.718

Table 231: Invariants for $M(S^3,(2,1),(3,1),(5,2))$, $S^3/P_{120}\times Z_{37}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$1-q^2+q^3$	=0.382	0	=0.000	0	=0.000	0.053
6	1	=1.000	0	=0.000	0	=0.000	0.083
7	$5+2q^2-2q^5$	=7.494	0	=0.000	0	=0.000	0.403

Table 232: Invariants for $M(S^3,(2,1),(3,1),(4,1))$, $S^3/P_{48}\times Z_{13}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0	=0.000	0.854
5	1	=1.000	-1	=−1.000	0	=0.000	0.000
6	10	=10.000	0	=0.000	0	=0.000	0.833
7	$4+2q^2-2q^5$	=6.494	$-4-2q^2+2q^5$	=−6.494	0	=0.000	0.000

Table 233: Invariants for $M(S^3,(2,1),(3,2),(3,5))$, $S^3/P_{24}\times Z_{17}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	2	=2.000	0	=0.000	0	=0.000	0.108

Table 234: Invariants for $S^3/P_{72}\times Z_5$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	0	=0.000	0	=0.000	0.543

Table 235: Invariants for $S^3/P_{24}\times Z_{13}$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$4+q^2-q^3$	=4.618	0	=0.000	0	=0.000	0.638
6	3	=3.000	0	=0.000	0	=0.000	0.250
7	$6+4q^2-2q^3+2q^4-4q^5$	=10.098	0	=0.000	0	=0.000	0.543

Table 236: Invariants for $M(S^3,(3,2),(3,2),(3,2))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	-1	=−1.000	0	=0.000	0.000
4	2	=2.000	$q-q^3$	=1.414	0	=0.000	0.854
5	$4+q^2-q^3$	=4.618	$-4-q^2+q^3$	=−4.618	0	=0.000	0.000
6	12	=12.000	$-12q+6q^3$	=−10.392	0	=0.000	0.134
7	$6+3q^2-2q^3+2q^4-3q^5$	=8.851	$-6-3q^2+2q^3-2q^4+3q^5$	=−8.851	0	=0.000	0.000

Table 237: Invariants for $M(S^3,(3,2),(3,2),(3,1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$3+q^2-q^3$	=3.618	0	=0.000	0	=0.000	0.500
6	9	=9.000	0	=0.000	0	=0.000	0.750
7	$9+6q^2-2q^3+2q^4-6q^5$	=15.592	0	=0.000	0	=0.000	0.839

Table 238: Invariants for $M(S^3,(3,2),(3,1),(3,1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	1	=1.000	1.000
4	2	=2.000	0	=0.000	0	=0.000	0.500
5	$4+q^2-q^3$	=4.618	0	=0.000	$4+q^2-q^3$	=4.618	1.276
6	12	=12.000	0	=0.000	0	=0.000	1.000
7	$3+q^3-q^4$	=3.445	0	=0.000	$3+q^3-q^4$	=3.445	0.371

Table 239: Invariants for $M(S^3,(3,1),(3,1),(3,1))$

r	$TV_0(M)_q$		$TV_1(M)_q$		$TV_2(M)_q$		TV^*
3	1	=1.000	0	=0.000	0	=0.000	0.500
4	1	=1.000	0	=0.000	0	=0.000	0.250
5	$5+2q^2-2q^3$	=6.236	0	=0.000	0	=0.000	0.862
6	9	=9.000	0	=0.000	0	=0.000	0.750
7	$3+q^3-q^4$	=3.445	0	=0.000	0	=0.000	0.185

APPENDIX C
LINKING PAIRING FROM THE LINKING MATRIX OF RATIONAL DEHN SURGERY.

Linking pairing (see Section 4.4) is an important homological invariant. It keeps much more topological information then the first homology group alone. It is well known that if M is a rational homology 3-sphere obtained by Dehn surgery on an (integer) framed oriented link L with a linking matrix A, then the linking pairing $\mathcal{L}_M$ can be presented by the matrix $-A^{-1}$ (see, for instance, [GL]). The well known proofs are based on consideration of the 4-manifold obtained by attaching 2-handles to B^4 along $L \subset S^3 = \partial B^4$ according to the given framings. Thus, these proofs can not be naturally extended to rational surgery descriptions of 3-manifolds. In this appendix we generalize this result to rational surgery presentations of M. Our proof is elementary and is based on a few simple facts from linear algebra.

Let L be an oriented link in S^3 with n components. We will say that L is a *rational framed link* if every component K_i of L is equipped with a rational number $\frac{p_i}{q_i}$, with $p_i, q_i \in \mathbf{Z}$. Let V_i be a small tubular neighborhood of $K_i \in S^3$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $m_i \in \partial V_i$ be a meridian of V_i and $l_i \in \partial V_i$ be a longitude (i.e., a curve in ∂V_i homologically trivial in $S^3 - V_i$). Orient l_i with respect to the orientation of K_i, and orient m_i so that the orientation of the pair (m_i, K_i) is positive. By *rational Dehn surgery* on the link L we mean the 3-manifold M obtained in the following way: remove the interiors of the tori $V_1, \dots, V_n$ from S^3 and attach n 2-handles $D_1 \times I, \dots D_n \times I$, where $I = [0, 1]$, such that $\partial D_i \subset \partial V_i$, $alg(\partial D_i, l_i) = p_i$, and $alg(\partial D_i, m_i) = q_i$, and after that glue in n 3-handles B^3 (see [Ro] for more details). In particular, $[\partial D_i] = p_i[m_i] + q_i[l_i]$ in $H_1(M)$. Note that if all q_i's are equal to ± 1, then we get an integer Dehn surgery (cf. Chapter 2).

The linking matrix $B = (b_{ij})_{n\times n}$ associated to L is defined as follows. The off-diagonal entries are the linking numbers $b_{ij} = lk(K_i, K_j)$, while the diagonal entries are the surgery coefficients $b_{ii} = \frac{p_i}{q_i}$. Let Q be the diagonal matrix $Q = (q_1) \oplus \dots \oplus (q_n)$. Then the integer matrix $\bar{B} = QB$ is the presentation matrix for $H_1(M)$ in the generators $[m_1], \dots, [m_n]$ (see, for instance, [Ho]).

THEOREM C.1. *Let M be a rational homology 3-sphere, then the linking pairing form of M in the generators $[m_1], \dots, [m_n] \in H_1(M)$ is given by the matrix $\Lambda^m = -B^{-1}$.*

PROOF. For every component K_i of L fix a Seifert surface S_i, i.e., an oriented surface embedded into $S^3 - K_i$ with $\partial S_i = K_i$. For every $i = 1, \dots, n$ define a 2-cycle $B_i \in C_2(M)$ by

$$B_i = (-D_i) \cup q_i \left(S_i - (S_i \cap (\bigcup_{j=1}^{n} V_j)) \right),$$

where $-D_i$ means the 2-cycle defined by D_i but with the opposite orientation, that is with the orientation opposite to the orientation of l_i. It is easy to check that

$$[\partial B_i] = -p_i[m_i] - q_i \sum_{j\neq i} lk(K_i, K_j)[m_j]$$

in $H_1(M)$. In other words, every B_i represents a relation in $H_1(M)$ defined by the ith row of the matrix $\bar{B}$, and we can write

$$-\bar{B}\begin{pmatrix}[m_1]\\ \vdots \\ [m_n]\end{pmatrix} = \begin{pmatrix}[\partial B_1]\\ \vdots \\ [\partial B_n]\end{pmatrix}.$$

Since M is a rational homology 3-sphere, $\bar{B}$ is a nondegenerate integer matrix. Therefore, there are unimodular integer matrices C and R such that

$$C\bar{B}R = T = \begin{pmatrix} t_1 & 0 & \cdots & 0\\ 0 & t_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & t_n\end{pmatrix}$$

with $t_i \in \mathbf{Z}_+$, $i = 1, \ldots, n$. This fact is essentially equivalent to the fundamental theorem of finitely generated abelian groups.

Therefore,

$$TR^{-1}\begin{pmatrix}[m_1]\\ \vdots \\ [m_n]\end{pmatrix} = -C\begin{pmatrix}[\partial B_1]\\ \vdots \\ [\partial B_n]\end{pmatrix}. \tag{C.1}$$

The matrix R^{-1} is integer, therefore we can consider a new generating set for $C_1(M)$

$$\begin{pmatrix}h_1\\ \vdots \\ h_n\end{pmatrix} = R^{-1}\begin{pmatrix}m_1\\ \vdots \\ m_n\end{pmatrix}.$$

The equation (C.1) implies that every 1-cycle $t_i h_i$ is homologically equivalent to 0 in M.

Now we will find the matrix Λ^h of the linking pairing $\mathcal{L}_M$ in the generators $[h_1], \ldots, [h_n]$. Let $R^{-1} = (\widetilde{r}_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$. Notice that $m_s \cdot B_k = alg(m_s, B_k) =$

$\delta_s^k q_s$, where $\delta_s^k = 1$ if $k = s$ and 0 otherwise, $k, s = 1, \dots n$. Then

$$\mathcal{L}_M([h_i],[h_j]) = \frac{h_i \cdot (-\sum_{k=1}^n c_{jk} B_k)}{t_j} = -\frac{1}{t_j} alg\left(\sum_{s=1}^n \widetilde{r}_{is} m_s, \sum_{k=1}^n c_{jk} B_k\right) =$$

$$= -\frac{1}{t_j} \sum_{s=1}^n \widetilde{r}_{is} c_{js} q_s.$$

Therefore, $\Lambda^h = -R^{-1} Q C^t T^{-1}$. Since $\begin{pmatrix} [m_1] \\ \vdots \\ [m_n] \end{pmatrix} = R \begin{pmatrix} [h_1] \\ \vdots \\ [h_n] \end{pmatrix}$, we have

$$\Lambda^m = R \Lambda^h R^t = -Q C^t T^{-1} R^t.$$

It remains to notice that

$$T^{-1} = R^{-1} \bar{B}^{-1} C^{-1} = (C^t)^{-1} (\bar{B}^t)^{-1} (R^t)^{-1}.$$

Thus,

$$\Lambda^m = -Q(\bar{B}^t)^{-1} = -Q(B^t Q)^{-1} = -B^{-1}.$$

The last equation follows from the fact that B is symmetric. $\square$

REFERENCES

[AM] A. Adem, R. J. Milgram, *Cohomology of finite groups*, Springer-Verlag, Berlin Heidelberg, Vol. 309 (1994).

[B] C. Blanchet, *Refined Quantum Invariants for Three-Manifolds with Structure*, Knot'95, Banach Center Publications 42, Warsaw, 1998, pp 11–22.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Remarks on the three-manifold invariants θ_p*, 'Operator Algebras, Mathematical Physics, and Low Dimensional Topology' (Nato Workshop July 1991), Vol. 5, 39–59.

[BJN] P. B. Bhattacharya, S. K. Jain, S. R. Nagpaul, *Basic abstract algebra*, Cambridge University Press, 1994.

[Br] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, Vol. 78, Springer (1982).

[Ca] B. G. Casler, *An imbedding theorem for connected 3-manifolds with boundary*, Math. Soc. 16 (1965), 559–566.

[Da] Philip J. Davis, *Circulant Matrices*, A Wiley-Interscience publication, 1979.

[De] F. Deloup, *Linking forms, Reciprocity for Gauss sums and invariants of 3-manifolds*, Trans. Amer. Math. Soc. (1998), to appear.

[DG] F. Deloup, C. Gille, *Abelian quantum invariants ideed classify linking pairings*, Preprint.

[DW] R. Dijkgraaf, E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys. 129 (1990), 393–429.

[Fe] M. Ferhane, *Exemples D'Invariants Quantiques et de Théorie Quantique des Champs Topologique en Dimension 2+1*, Thése De Doctorat, Université de Nantes, 1996 (French).

[FG] G. Felder and O. Grandjean, *On combinatorial three-manifold invariants*, Low Dimensional Topology and Quantum Field Theory, Edited by H. Osborn, Plemen Press, New York, 1993, pp. 31–50.

[Fo] A. T. Fomenko, *Visual geometry and topology*, Moscow University, 1992 (in Russian).

[FR] R. Fenn, C. Rourke, *On Kirby's calculus of links*, Topology **18**, 1–15 (1979).

[G] P. Gilmer, *Quantum Invariants of Periodic Three-Manifolds*, Preprint, math.GT/9902122, 1999.

[GL] C. McA. Gordon, R. A. Litherland, *On the signature of a link*, Inventiones math., 47 (1978), 53–69.

[Ho] J. Hoste, *A formula for Casson's invariant*, Transactions of the American Math. Soc., Vol. 297, No. 2 (1986), 547–562.

[Je] L. Jeffrey, *Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation*, Commun. Math. Phys. N 147, 563–604 (1992).

[K] R. C. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45**, 35–56 (1978).

[Ka] U. Kaiser, *Link theory in manifolds*, Springer, 1997 (lecture notes in mathematics; 1669).

[KB-Pr] J. Kania-Bartoszyńska, J. H. Przytycki, *3-manifold invariants and periodicity of homology spheres*, Preprint, 1997.

[KL1] L. H. Kauffman and S. Lins, *Computing Turaev–Viro invariants for 3-manifolds*, Manuscripta Math. 72 (1991), 81–94.

[KL2] L. H. Kauffman and S. Lins, *Temperley–Lieb recoupling theory and invariants of 3-manifolds*, Princeton University Press, Princeton, N. J., 1994.

[KM] R. C. Kirby, P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin–Turaev for $sl(2, \mathbf{C})$*, Inv. Math. 105 (1991), 473–545.

[L] S. Lang, *Algebraic number theory*, Springer GTM110, 1986.

[Li1] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math., 76, 1962, 531-538.

[Li2] W. B. R. Lickorish, *Calculations with the Temperley-Lieb algebra*, Comm. Math. Helv. 67 (1992), 571–591.

[Li3] W. B. R. Lickorish, *Three-manifold invariants and the Temperley-Lieb algebra*, Math. Ann. 290 (1991), 657–670.

[Li4] W. B. R. Lickorish, *The skein method for 3-manifold invariants*, J. Knot Th. Ram. 2 (1993), 171–194.

[LL] Bang-He Li and Tian-Jun Li, *Generalized Gaussian sums and Chern-Simons-Witten-Jones invariants of lens spaces*, J. of Knot Th. and Ram., Vol. 5, No. 2 (1996), 183–224.

[M1] S. V. Matveev, *Transformations of special spines and the Zeeman conjecture*, Math. USSR Izvestia, Vol. 31, 2, 1988, 423–434.

[M2] S. V. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Applicandae Mathematicae. 19(1990), 101–130.

[M3] S. V. Matveev, *Tables of 3-manifolds up to complexity 6*, Max-Planck-Institut für Mathematik Preprint Series 1998(67), 50 p.

[MB] J. W. Morgan, H. Bass, eds., *The Smith conjecture*, Pure Appl. Math. **112**, New York Academic Press (1984).

[MOO] H. Murakami, T. Ohtsuki and M. Okada, *Invariants of Three-Manifolds Derived from Linking Matrices of Framed Links*, Osaka J. Math., 29 (1992), 545–572.

[MR] G. Masbaum, J. Roberts, *A simple proof of integrality of quantum invariants at prime roots of unity*, Math. Proc. Cambridge Phil. Soc., 121, 443–454 (1997).

[Mu1] H. Murakami, *Quantum* $SU(2)$*-invariants dominate Casson's* $SU(2)$*-invariant*. Math. Proc. Cambridge Philos. Soc. **115**, 253–281 (1994).

[Mu2] H. Murakami, *Quantum* $SO(3)$*-invariants dominate the* $SU(2)$*-invariant of Casson and Walker*, Math. Proc. Cambridge Philos. Soc., 117 (1995), no. 2, 237–249.

[N] J. R. Neil, *Combinatorial calculation of the various normalizations of the Witten invariants for 3-manifolds*, J. of Knot Theory and Its Ramifications, Vol. 1, 4 (1992), 407–449.

[O1] T. Ohtsuki, *A polynomial invariant of integral homology 3-spheres*, Math. Proc. Camb. Phil. Soc., **117**, pp 83–112, (1995).

[O2] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres*, J. of Knot Th. and Ramif. 5 (1996), 101–115.

[Ov] M. Ovchinnikov, *Special spines of Waldhausen manifolds*, Kandidatskaya dissertation, Chelyabinsk State University, 2000 (Russian).

[Pr1] J. H. Przytycki, *Algebraic topology based on knots: an introduction*, Proceedings of Knots 96, edited by S. Suzuki, 1997, World Scientific Publishing Co., pp 279–298.

[Pr2] J. H. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe Math. J., 16(1), 1999, 45-66.

[Pr3] J. H. Przytycki, *Cyclic actions on S^2- and P^2-bundles over S^1*, Colloquium Mathematicum, Vol. XLVII, Fasc. 2 (1982), 241–254.

[Pr4] J. H. Przytycki, *A q-analogue of the first homology group of a 3-manifold*, Perspectives on quantization (South Hadley, MA, 1996), 135–144, Contemp. Math., 214, Amer. Math. Soc., Providence, RI, 1998.

[Ro] Dale Rolfsen, *Knots and Links*, Publish or Perish, Inc., 439 p. (1976).

[RT] N. Reshetikhin, V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent., Math. 103 (1991), 547-597.

[Si] Adam Sikora, *Cyclic Group Actions on Manifolds*, PhD Preliminary Oral Examination, U. of Maryland in College Park, November 1999.

[Sp] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Company, 1966.

[T1] V. G. Turaev, *Topology of shadows*. Preprint, 1991.

[T2] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, Walter de Gruyter, Berlin, New York, 1994.

[Th] W. Thurston, *The geometry and topology of three-manifolds*, Lecture Notes, Princeton Univ. (1979).

[TV] V. G. Turaev and O. Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology, 31 (1992), 866–902.

[W] K. Walker, *On Witten's 3-manifold invariants*, Preprint, 1991.

[Wa] A.D.Wallace, *Modifications and cobounding manifolds*, Can. J. Math., 12, 1960, 503-528.

[Wak] M. Wakui, *On Dijkgraaf-Witten invariant for 3-manifolds*, Osaka J. Math. 29 (1992), pp. 675–696.

[Wi] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Comm. Math. Phys. 121, 1998, pp 351–399.

[Ya] S. Yamada, *The absolute value of the Chern-Simons-Witten invariants of lens spaces*, J. of Knot Theory and Ramif. Vol. 4, N 2. (1995) 319–327.

[Yo] Y. Yokota, *Skeins and quantum $SU(N)$ invariants of 3-manifolds*, Math. Ann. 307 (1997), 109–138.

Papers by the Author:

[MOS] S. V. Matveev, M. A. Ovchinnikov, M. V. Sokolov, *Construction and properties of the t-invariant*, (to appear in Proceedings of the conference "Topology and Dynamics: Rokhlin memorial", POMI, St. Petersburg, 2000).

[PrS] J. H. Przytycki, M. Sokolov *Surgeries on periodic links and homology of periodic 3-manifolds*, Preprint, 2000.

[S1] M. Sokolov, *On the absolute value of the SO(3)-invariant and other summands of the Turaev-Viro invariants*, Knot'95, Banach Center Publications 42, Warsaw, 1998, pp 310-321.

[S2] M. Sokolov, *Which lens spaces are distinguished by the Turaev-Viro invariants?*, Mat. Zametki, Vol. 61, No.3 (1997), pp 468-470.

[S3] M. Sokolov, *The Turaev-Viro invariant is a sum of three invariants*, Canadian Math. Bull., Vol. 39 (4), 1996, pp 468-475.

[S4] M. Sokolov, *On Turaev-Viro invariants for 3-manifolds*, VINITI preprint 583-B93 (Russian).

[S5] M. Sokolov, *Calculation of Turaev-Viro invariants for 3-manifolds and solution of Kauffman-Lins conjecture*, Theses of XXX International Scientific Student Conference. Novosibirsk, 1992, pp 82-88 (Russian)

[S6] M. Sokolov, *On knot move "Maltese Cross"*, Materials of Scientific Students Conference. Chelyabinsk, 1990, p. 49 (Russian).

Talks at Conferences:

[S7] *Surgeries on periodic links and homology of periodic 3-manifolds*, (Joint work with J. Przytycki) Knots in Washington X, Washington, DC, January 29, 2000.

[S8] *Deloup invariants, Dichromatic skein module, and periodic 3-manifolds*, Joint Mathematics Meeting in Washington, DC in January 2000, Special Session on Invariants of Knots and 3-Manifolds.

[S9] *Murakami-Otsuki-Okada invariants and periodicity of 3-manifolds*, AMS meeting in Las Vegas, USA, Special Session on Symmetries of Knots and 3-Manifolds, April 10, 1999.

[S10] *Invariants of 3-manifolds obtained using 2-term skein relations*, Knots in Hellas'98; Delphi, Greece, August, 1998.

[S11] *Which lens spaces can be distinguished by the absolute values of the Witten-Reshetikhin-Turaev invariants*, Knots'96; Tokyo, Japan, July, 1996.

[S12] *The Turaev-Viro invariant as a sum of three invariants*, Mini-semester on Knot theory; Banach center, Warsaw, Poland, July-August 1995.

[S13] *The Turaev-Viro invariant as a sum of three invariants and invariants of Turaev-Viro type*, Kiev International Topology Conference; Kiev, Ukraine, June, 1995.

[S14] *Calculations of the Turaev-Viro invariants*, 30th International Scientific Student Conference in Novosibirsk; Novosibirsk State University, Russia, April, 1992.

[S15] *On knot move "Maltese cross"*, Scientific Student Conference in Krasnoyarsk; Krasnoyarsk State University, Russia, March, 1991.